Fraction of Nonnegative Polynomials which are Sums of Squares

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Introduction

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- Parrilo created an algorithm to optimize SOS polynomials in polynomial time via semidefinite programming. Polynomial optimization has applications in many areas such as electrical engineering.
- All SOS polynomials are nonnegative. How many nonnegative polynomials are SOS?

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- For nonnegative polynomials of fixed degree, previous results by Blekherman show that the fraction of nonnegative polynomials that are SOS approaches zero as the number of variables increases.
- What about polynomials in few variables of low degree?

Cone of Polynomials

- Focus on bivariate polynomials $f(x, y)$, $\deg_y(f)$ and $\deg_x(f)$ at most 4:

$$f(x, y) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6y + c_7xy + c_8x^2y + c_9x^3y + c_{10}x^4y + c_{11}y^2 + c_{12}xy^2 + c_{13}x^2y^2 + c_{14}x^3y^2 + c_{15}x^4y^2 + c_{16}y^3 + c_{17}xy^3 + c_{18}x^2y^3 + c_{19}x^3y^3 + c_{20}x^4y^3 + c_{21}y^4 + c_{22}xy^4 + c_{23}x^2y^4 + c_{24}x^3y^4 + c_{25}x^4y^4$$

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- The set of nonnegative polynomials of this type form a 25 dimensional cone, and the set of sums of squares of polynomials form a cone inside.
- Intersect the cones with the hyperplane of polynomials

$$\int_{S^1 \times S^1} f \, \mathrm{d}\mu = 1.$$

Main Idea

- 24 dimensional convex body of sum of squares polynomials inside convex body of nonnegative polynomials.

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- Find ratio of the volumes to find the fraction.

Hit and Run

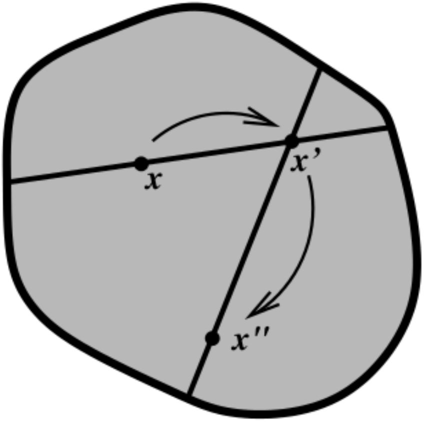

Figure: Hit and Run algorithm

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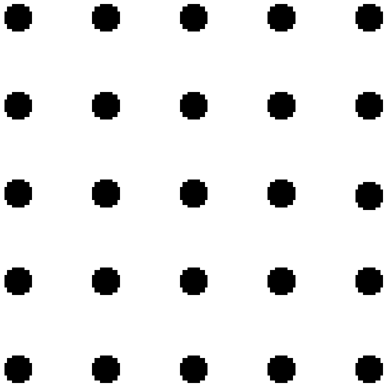
- How do we find the values of t at the endpoints?

The support A of a polynomial



A-discriminant

- Given a polynomial $h(x_1, \ldots, x_n)$ with support A, the A-discriminant $\Delta_A(h)$ is an irreducible polynomial in the coefficients of h which vanishes when h has a degenerate root (i.e. $\frac{\partial h}{\partial x_i} = 0$ for all i).

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- A nonnegative polynomial h is on the boundary of our cone when $\Delta_A(h) = 0$. However, Δ_A is not easy to compute!

Finding the values of t

- The resultant of polynomials $h_1, \ldots, h_k$ is an irreducible polynomial in the coefficients of $h_1, \ldots, h_k$ which vanishes when $h_1, \ldots, h_k$ have a common root.

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- The principal A-determinant E_A is the following resultant:

$$E_A(h) = Res_{(A,A,A)}(h, x\frac{\partial h}{\partial x}, y\frac{\partial h}{\partial y})$$

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$$E_A(h) = (\Delta_A(h))(\Delta\_\Delta\_\Delta_{|}\Delta_{|}\Delta_{.}\Delta_{.}\Delta_{.}\Delta_{.})$$
- To find the values of t at the endpoints, find the roots of $\Delta_A(f + t \cdot v)$ closest to 0!