OPTIMUM HARVESTING MODELS FOR FISHERY POPULATIONS

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ABSTRACT. Fishery management is the consideration of the ecological effects of harvesting. Fisherman work to provide fish for a growing human population but because of this some fish populations have been dangerously declining. It is important to balance ecological and economic needs. In this paper we investigate various deterministic models of fishery populations. A simple logistic model, a skewed logistic model with a quadratic term, and a model that demonstrates the Allee effect have all been considered with a constant harvest rate as well as time dependent havesting. Optimization and numerical calculations were used to determine the harvest rate that produces maximum yield under different population density scenerios.

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1. INTRODUCTION: A BRIEF OUTLINE

Fishery management is the balance between harvesting and its ecological implications. Worldwide, fish populations are in various states; some are healthy and some are depleated due to environmental and human factors. The population conditions greatly impact the dynamics of the system. In fishery management, it is important to fish in such a way that a species is sustainable and not in danger of becoming endangered or going extinct. We do not want to over harvest small fish populations. The key idea in our study was implementing different fishing strategies to maximize catch in a sustainable manner. We have construted three models to examine how harvest in such a way.

Species throughout the oceans currently have different population sizes; there are some populations above equilibrium and some below. These factors should be taken into consideration when constructing a harvest strategy. We have investigated numerous populaton conditions in working through our models to do so. Questions of interest included: how far can we harvest before putting a population that is currently healthy in danger? Can we still harvest populations that are a low population density? If so, what is the best way to harvest to maximize yield? We have answered these questions to find ways to make fisheries more efficient.

Thus given the rate of change of a fish population $\frac{dn}{dt}$ where $n(t)$ is population density, we are interested in maximizing yield without hurting the population. This important in commerical fishing for two reasons: i) economically we want the greatest catch to supply the demand for fish thus we maximize yield. Furthermore ii) ecologically, we do not want to deplete the population to keep diversity in the oceans and allow for fish to be harvested in the future. We have represented population density with $n > 0$, harvest rate $0 \leq f < 1$, and yield $Y = fn$.

2. Background

The three models worked with include a simple logistic model, a skewed logistic model with a quadratic term, and a model with the Allee effect. Previous work has done greatly with a logistic model for fish populations

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - qEn \tag{1}$$

where r is the intrisic population growth rate and is constant , K is carrying capacity of the environment without fishing, and n is population size [1]. Furthermore the harvest rate is qEn the product a constant that measures catchability q, effort E, and population size. The value qE is called fishing mortality [1].

Then by setting (1) equal to zero, there are two equilibria. These occur when the growth rate of the population is equivalent to the harvest rate ie. $rn\left(1 - \frac{n}{K}\right) = qEn$. First there is the equilibria

$$n^* = K\left(1 - \frac{qE}{r}\right) \tag{2}$$

the other is extinction where the fish population has been depleated. These equilibria are seen in Figure 1. When fishing mortality is small (2) is stable; if the population increases past n^*, harvest rate is greater than growth rate, the stock decreases back to equilibrium. If the population decreases past n^*, harvest is less than growth, the population increases to n^*. In this case, since the equilibrium defined in (2) is stable it is sustainable, then in turn extinction is unstable.

Consequently, as fishing mortality increases the equilibrium (2) decreases and shifts to the left (Figure 1). Eventually, for large enough values of fishing mortality, and n^* has gone far enough to the left that harvest exceeds growth for all positive population levels. Then $n^* = 0$ is the stable equilibrium.

Because the goal is to maximize yield, equilibrium (2) is considered. Thus the equation for sustainable yield is
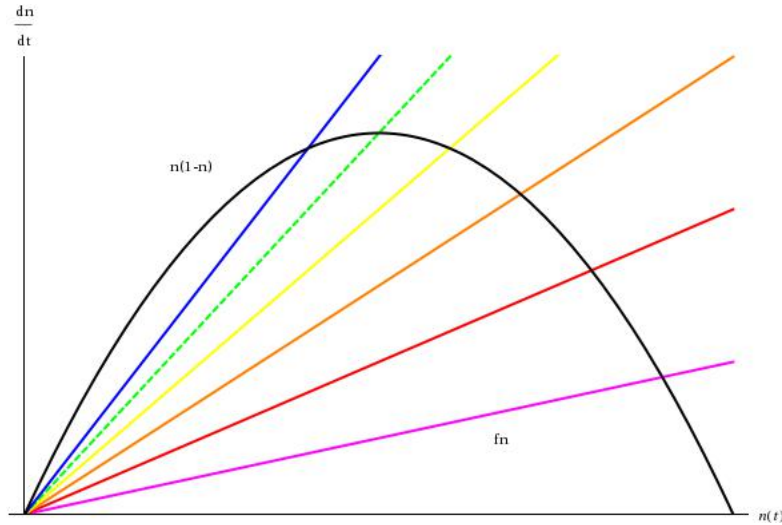
$$Y = qEn^* = qEK\left(1 - \frac{qE}{r}\right).$$

FIGURE 1. Equilibrium for our simplified logistic model which graphs intrinsic growth $n(1-n)$ and harvest fn for various $0 < f < 1$. Intersection points represent the equilibrium n^* where $\frac{dn}{dt} = 0$.

Suppose that catchability is fixed, since it is a constant, then increased fishing effort will increase sustainable yield - up to a point. Then maximum sustainable yield (MSY) occurs when

$$\frac{dY}{dE} = qK\left(1 - \frac{2qE}{r}\right) = 0$$

which by solving gives the correspoinding optimal level of effort

$$E_{MSY} = \frac{r}{2q}$$

and morever maximum sustainable yield

$$MSY = \frac{rK}{4}.$$

This model is simplified and we have done further research by varying harvest rate to maximize yield for populations that are not at equilibrium. We have also constructed other models and done similar processes using these models.

3. METHODS: SOLVING OUR MODELS

In all three models (simple logistic, skewed, and Allee) we first considered the population density n^*, harvest rate f^*, and maximum yield

$Y^* = f^*n^*$ at equilibrium. Note that since the population is at equilibrium, this is a sustainable way to harvest fish because the population is not endanger of being depleated. This is one possible fishing strategy that would sustain the population giving us a possible solution to the ecological problem.

However, since the key is to economically build up the commerical fishing industry, we investigated two main questions: i) current population levels are not necessarily at equilibrim; what is the best strategy to harvest under such a situation? ii) Population dynamics cannot always be represented by a simple logistic model such as the on [1] used so we constructed other population models; the skewed logistic model and Allee effect model. Furthermore under these models we investigated finding the best strategy when we incorperate the skew or Allee effect. We used optimization processes to find the constant harvest rate that maximizes yield over time T by taking the derivative of yield with respect to harvest rate. This is a partial derivative since yield is a function of population size n and harvest rate f where n is a function of time t and f. Since the fish catch of a given year is $Y = fn$, then the yield function over time T is defined as

$$Y(n, f, t) := \int_0^T f(t)n(t, f)dt.$$

This boiled down to an implicit formula that can be used to numerically calculate $n(t)$ as well as the optimum f that maximizes Y. This calculation was done for the Allee effect model and the skewed logistic model (Allee model with $a = 0$).

We then used MATLAB [2] to do our numerical calculations and determine the optimum harvest rate that maximizes yield all three models without using our implicit relationships. The MATLAB [2] coding used ODE 45 to to find the solution $n(t)$ to the differential equation $\frac{dn}{dt}$. Then this code used trial and error processes on $0 < f < 1$ to find the harvest rate that yields maximum catch by finding the zeroes of $\frac{dY}{df}$. This produced plots of yield as a function of harvest rate.

Additionally, we constructed a two stage harvesting which model step function and compared this strategy to the equilibrium harvesting and constant harvesting in terms of yield. This is the basis of our step function; suppose we want to consider the fish population over time T, let f_t be defined

as:

$$(3) \qquad f(t) = \begin{cases} f_1, & t \in [0, \frac{T}{2}] \\ f_2, & t \in (\frac{T}{2}, T] \end{cases}$$

where $f_1 \neq f_2 \geq 0$ are constant rates of fishing. This means we first harvest at a rate of f_1 and then f_2 so our population density n depends on time t as well as harvest rates f_1 and f_2 ie. $n = n(t; f_1, f_2)$.

We examined our three models under this time dependent harvest rate and compared our possible yields to a constant harvest rate. This was done for initial conditions of different population values. The initial conditions of the fishery population are very important when determining harvest rates. Let $n(0) = n_0$ be the initial population density. We are interested in how harvest strategies that increase the average maximum yield depend on the initial population so we have varied initial conditions to understand how maximum harvest rate depends on these conditions.

4. RESULTS

4.1. **A Simple Logistic Model.** We began our research with a single stage simple logistic model

$$(4) \qquad \frac{dn}{dt} = n(1 - n) - fn$$

where n is the population density and f is fishing rate.

By solving (4) at equilibrium (ie. $\frac{dn}{dt} = 0$) we have the solution $n^* = 1 - f^*$ and $Y^* = f^*(1 - f^*)$ and moreover $n^* = \frac{1}{2}$, $f^* = \frac{1}{2}$, $Y^* = \frac{1}{4}$ (Figure 2). This agrees with the calculations done on the logsitic model as previously described by Kot [1].

We then consider solutions away from equilibrium that will still sustain the population. We used MATLAB [2] to find the harvest rate that maximizes yield over time of $T = 5, 10, 15, 20$ units and plotted yield as a function of harvest (Figure 4). We also found yield as a function of harvest for various initial conditions $n_0 = 1, 2, 3$ to determine how the initial conditions impact the population dynamics (Figure 5).
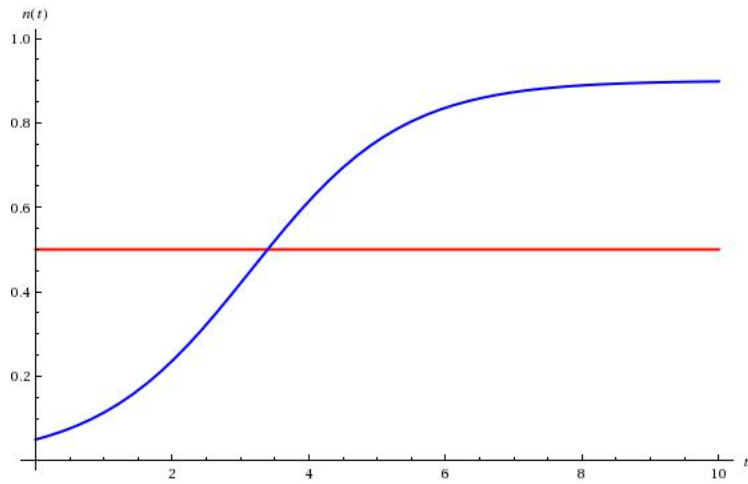
FIGURE 2. Solution $n(t)$ of simple logistic differential equation over time t with equilibrium value $n^* = \frac{1}{2}$.
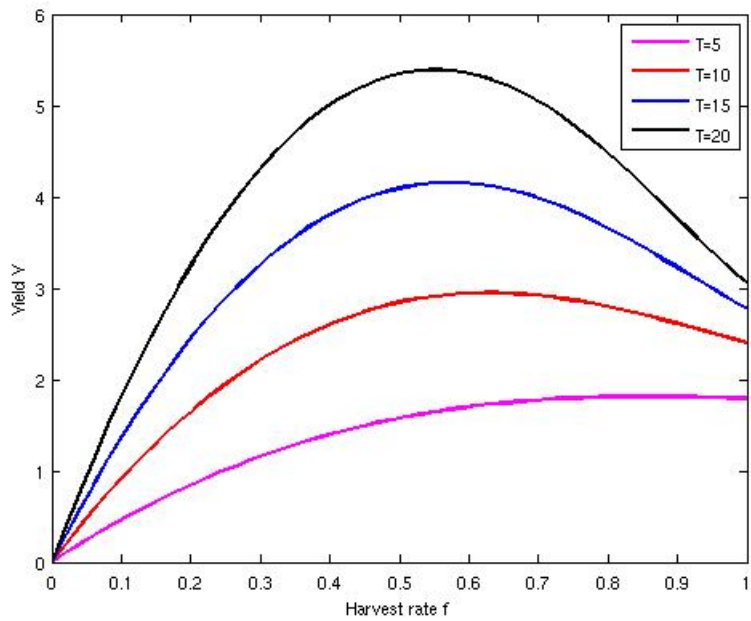

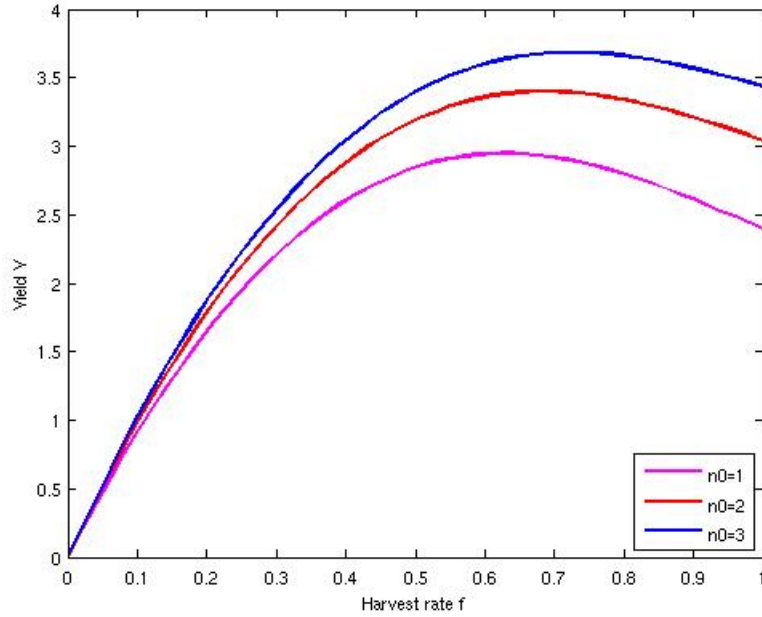FIGURE 3. Plot of yield as a function of harvest rate f over $T = 5, 10, 15, 20$.

FIGURE 4. Plot of yield as a function of harvest rate f for initial populations of $n_0 = 1, 2, 3$ over time $T = 10$.

4.2. **A Skewed Logistic Model.** To factor in a penalty on reproduction, we first used the following model

$$\frac{dn}{dt} = n^2(1 - n) - fn$$

so we get $f^* = -(n^*)^2 + n^*$ and the yield function

$$Y^* = -(n^*)^3 + (n^*)^2$$

which once solved gives the equilibrium population density $n^* = \frac{2}{3}$, harvest rate $f^* = \frac{2}{9}$, and yield $Y^* = \frac{4}{27}$ (Figure 5).

Now we investigated sustainable solutions that maximize yield; we determined the optimum harvest rate that maximizes yield over time of $T = 5, 10, 15, 20$ units and plotted yield as a function of harvest (Figure 6). We repeated this process for various initial conditions $n_0 = 1, 2, 3$ to determine how the initial conditions impact the population dynamics (Figure 7).
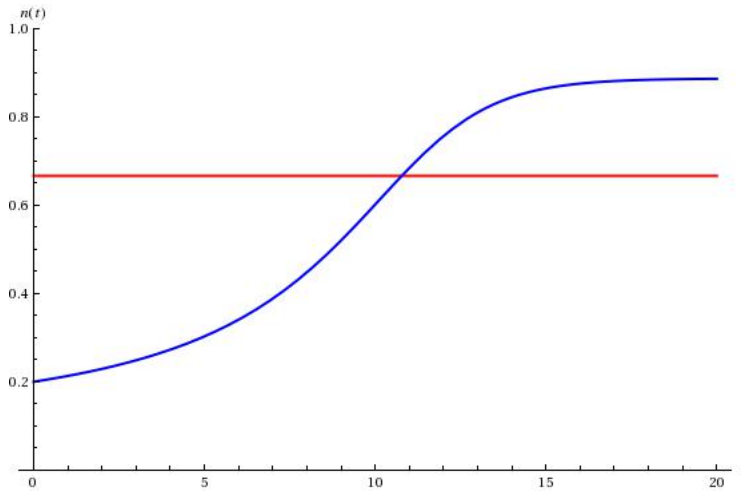
FIGURE 5. Solution $n(t)$ of skewed logistic model $\frac{dn}{dt} = n^2(1-n) - fn$ with equilibrium population density $n^* = \frac{2}{3}$ and harvest rate $f^* = \frac{2}{9}$.
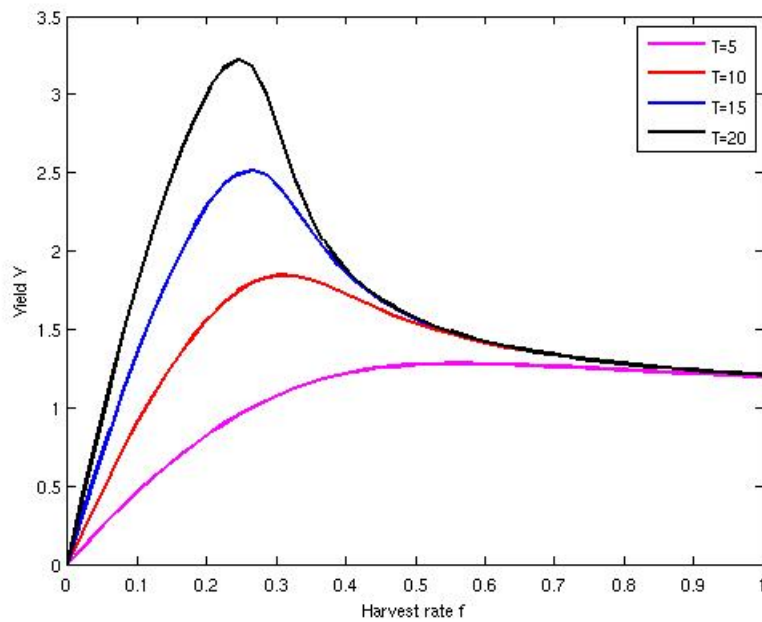

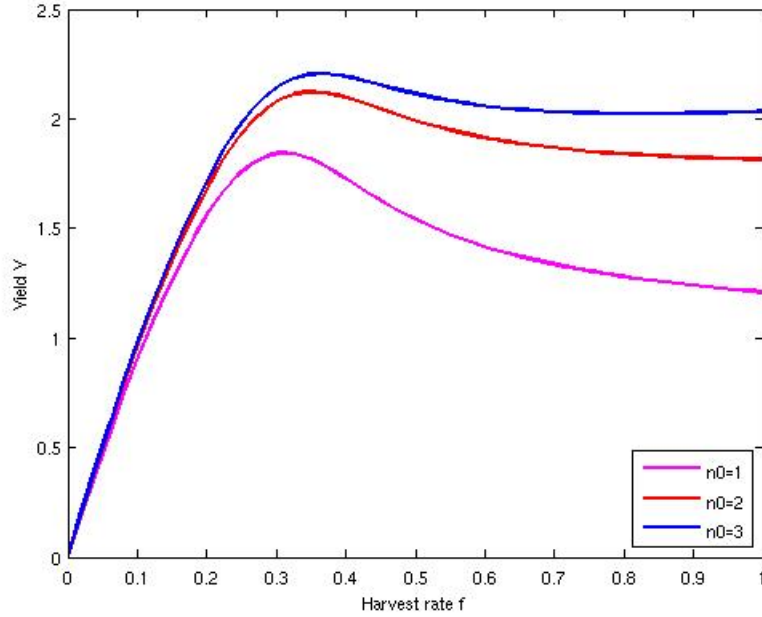FIGURE 6. Yield as a function of harvest f for the skewed logistic model over time $T = 5, 10, 15, 20$.

FIGURE 7. Yield as a function of harvest for initial population sizes $n_0 = 1, 2, 3$ over time $T = 10$.

4.3. **A Model with the Allee Effect.** Let a denote the Allee effect where $0 < a < 1$. An Allee effect occurs when there is a positive correlation between population density and reproductive rate. This happens because it is more difficult for the fish to find mates. We have modeled this phenomenon as follows

$$\frac{dn}{dt} = n(n-a)(1-n) - fn.$$

The added paramater $0 < a < 1$ vastly changes the population density over time (Figure 8). Based on the differential equation, at equilibrium we have

$$0 = (n^*)^2 + (1+a)n^* - (f^* + a)$$

and solving this with the quadratic formula we get the equilibrium population density of

$$n^* = \frac{-1 - a \pm \sqrt{a^2 + 6a + 4f^* + 1}}{2}.$$

This means that there are two equilibria, n_1^* and n_2^* (Figure 9).

Now let us consider the plus case where $n_1^* = \frac{-1-a+\sqrt{a^2+6a+4f^*+1}}{2}$. Then by means of implicit differentiation and optimization we get

$$f_1^* = \frac{1}{9}\left[-1 - 8a - a^2 + \sqrt{a^4 + 7a^3 + 12a^2 + 7a + 1}\right]$$

$$Y_1^* = -\frac{1}{18}\left[1 + a(8+a) - \sqrt{(1+a)^2(1+5a+a^2)}\right] \times \left[-1 - a + \sqrt{1+6a+a^2+4f^*}\right].$$

This solution is unstable because when small disturbances that change the population n the population goes extinct rather than returning to the equilibrium n_1^*.

For the minus case, ie. $n_2^* = \frac{-1-a-\sqrt{a^2+6a+4f^*+1}}{2}$, we use a parallel argument and get another equilibrium with the following solution

$$n_2^* = \frac{-1 - a - \sqrt{a^2 + 6a + 4\overline{f} + 1}}{2}$$

$$f_2^* = \frac{1}{9}\left[-1 - 8a - a^2 - \sqrt{a^4 + 7a^3 + 1a^2 + 7a + 1}\right]$$

$$Y_2^* = \frac{1}{18}\left[1 + 8a + a^2 + \sqrt{(1+a)^2(1+5a+a^2)}\right] \times \left[1 + a + \sqrt{1+6a+a^2+4f}\right].$$

This solution is stable since after small disturbances the population density returns to equilibrium n_2^*.

We then did the calculations to determine the implicit relationship between $n(t)$ and the optimum harvest rate f that maximizes yield in a sustainable manner; our calculations follow

$$\frac{dn}{dt} = n(n - a)(1 - n) - fn$$

Then we determined

(5) $$\frac{\frac{dn}{dt}}{n(t)[(n - a)(n - 1) + f]} = -1$$

Based on the differential equation, we require $f < \left(\frac{1-a}{2}\right)^2$ so the population does not decrease to extinction. Then

$$(n - a)(n - a) + f = \left(n - \left(\frac{1+a}{2}\right)\right)^2 - \left(\left(\frac{1-a}{2}\right)^2 - f\right)$$

$$= (n - \alpha(f))(n - \beta(f))$$

with $\alpha(f) := \frac{1+a}{2} + \sqrt{\left(\frac{1-a}{2}\right)^2 - f} > 0$ and $\beta(f) := \frac{1+a}{2} - \sqrt{\left(\frac{1-a}{2}\right)^2 - f} > 0$.

We split this up using partial fractions to get

$$\frac{1}{n(n-\alpha)(n-\beta)} = \frac{A}{n} + \frac{B}{n-\alpha} + \frac{C}{n-\beta}$$

with $A = \frac{1}{\alpha\beta}, B = \frac{-1}{\alpha(\beta-\alpha)}$ and $C = \frac{1}{\beta(\beta-\alpha)}$. Since fish yield over time T is

$$Y = \int_0^T fn(t,f)dt$$

we want to find $\frac{dY}{df}$ to maximize yield. So we have

(6)
$$\frac{d}{df}Y = \int_0^T \left[n(t,f) + f\partial_f n(t,f) \right] dt = 0$$

and calculated $\partial_f n(t,f)$ from the differential equation (5). We integrated to determine the following implicit relationship

$$-T = -\int_0^T dt = A\log(n) + B\log(n-\alpha) + C\log(n-\beta) - [A\log(n_0) + B\log(n_0 - \alpha) + C\log(n_0 - \beta)]$$

and defined $\mathfrak{F}(n((t,f),f)) := A\log(n) + B\log(n-\alpha) + C\log(n-\beta)$. Then the implicit formula arose

(7)
$$\mathfrak{F}(n) = \mathfrak{F}(n_0) - T$$

so we used differentiated (7) with respect to f to find $\partial_f n$ and have

(8)
$$\frac{d}{df}\mathfrak{F}(n(t,f),t)) = \mathfrak{F}_{,n}\left(n(t,f)\right)\partial_f n(t,f) + \mathfrak{F}_{,f}\left(n(t,f),f\right)$$

with $\mathfrak{F}_{,n}\left(n,f\right) := \partial_n \mathfrak{F}(n,f)$ and $\mathfrak{F}_{,f}\left(n,f\right) := \partial_f \mathfrak{F}(n,f)$ where n is treated as a variable independent of f. Then we solved $\frac{d}{df}\mathfrak{F}(n(t,f),t)) = 0$ (8) to get $\partial_f n$ so we have

$$\partial_f n = \frac{\mathfrak{F}_{,f}\left(n_0,f\right) - \mathfrak{F}_{,f}\left(n(t,f),f\right)}{\mathfrak{F}_{,n}\left(n(t,f),f\right)}.$$

We can plug this back into the yield differential equation (6) and find the optimal harvest rate that maximizes yield. Thus we have an algorithim to calculate the optimum harvest rate.

Similiarly we an do this numerically in MATLAB [2]. We considered the Allee effect parameter $a = 0.1$; as $0 < a < 1$ increases there is more penalty on reproduction. From the differential equation with $a = 0.1$, we determined sustainable population density solutions away from equilibrium and then found the harvest rate that maximizes yield over time of $T = 5, 10, 15, 20$ units (Figure 10). We also found the yield function for various initial conditions $n_0 = 1, 2, 3$ to determine how the initial conditions impact the population dynamics (Figure 11).
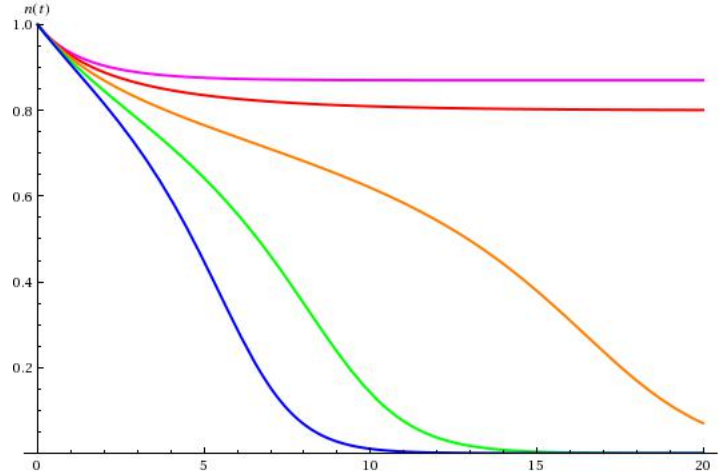

FIGURE 8. Population density solution to the Allee model differential equation $\frac{dn}{dt} = n(n-a)(1-n) - fn$ for Allee effect $a = 0.1$ (pink), $a = 0.3$ (red), $a = 0.5$ (orange), $a = 0.7$ (green), and $a = 0.9$ (blue) for harvest rate $f = 0.1$.
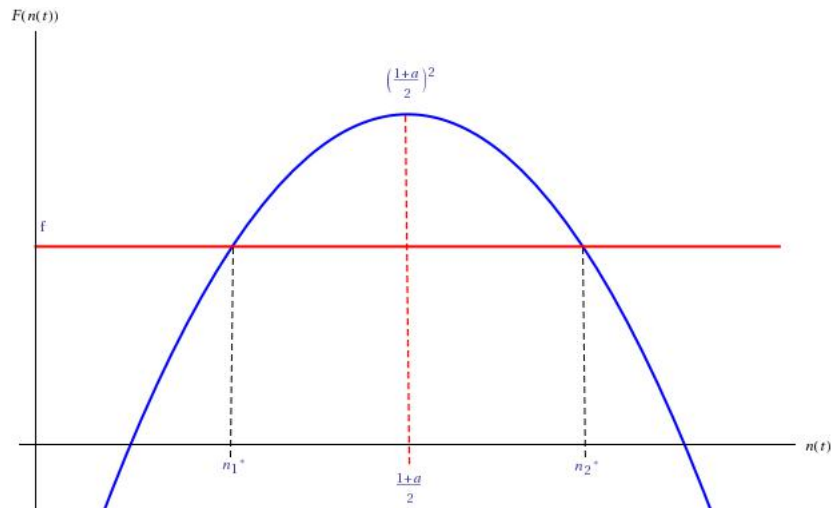

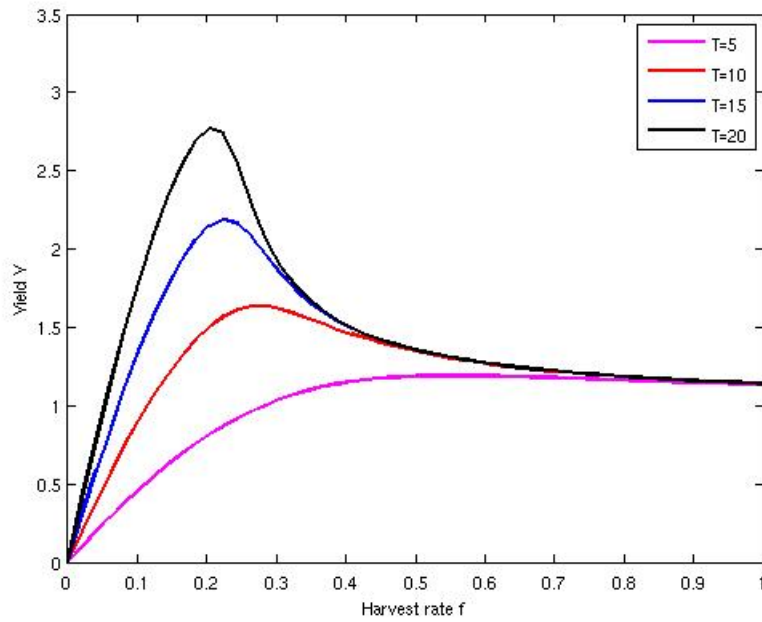FIGURE 9. Plot of the unstable equilibrium n_1^* and the stable n_2^* equilibrium with harvest rate f.

FIGURE 10. Yield as a function of harvest for $a = 0.1$ over time $T = 5, 10, 15, 20$. Initial population size of $n_0 = 1$.
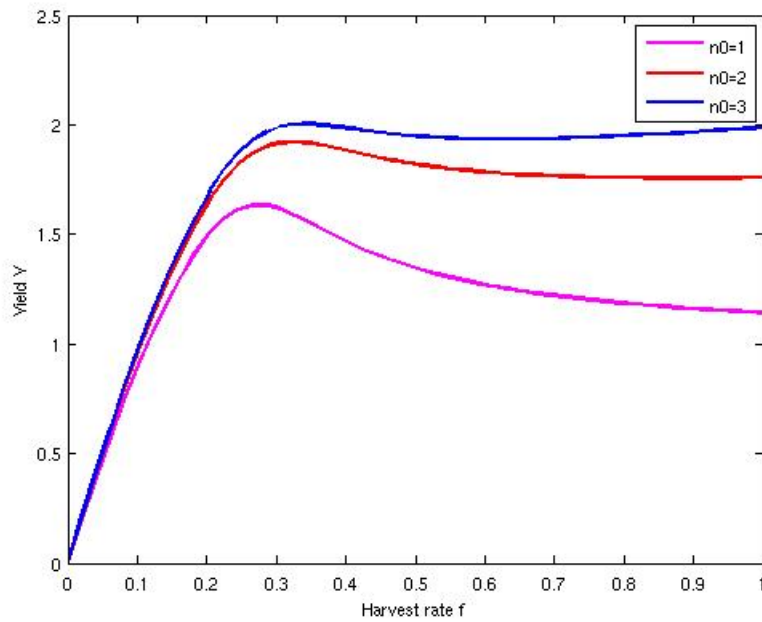

FIGURE 11. Yield for various initial conditions $n_0 = 1, 2, 3$ over time $T = 10$ with Allee effect parameter $a = 0.1$.

4.4. A Two Stage Harvesting Strategy.

Recall the two stage harvesting strategy (9)

$$(9) \qquad f(t) = \begin{cases} f_1 & t \in [0, \frac{T}{2} \\ f_2 & t \in (\frac{T}{2}, T] \end{cases}$$

By using numerical solving methods in Mathematica we found that this two stage harvesting strategy will increase yield in populations with initial conditions below the stable equilibrium. This was done with various values for f_1 and f_2.

5. ANALYSIS

In all three models, when the intial population is below the stable equilibrium, time dependent harvesting strategies increase the maximum average yield (ie. number of fish per time). This means that unhealthy populations can still be fished while still sustaining the population size, and we can do so in a manner that increases yield. If the population is initially above the equilibrium value then there is no difference between constant and time dependent harvesting strategies.

The model used to demonstrate fishery population growth impacts the yield of fish caught. This is shown in our various plots of yield vs. harvest rate. Furthermore our models have different optimum harvest rates; the model used impacts the population and harvesting dynamics greatly.

A simple logistic model does not penalize reproduction for low population levels. In theory, maximum yield could be obtained by waiting for the population to reach carrying capacity and implementing harvesting strategies. To include a reproductive penalty the skewed logistic model uses a square term. Further penalty is incorportated through the model exhibiting the Allee effect. When the Allee parameter increases the yield decreases which is demonstrated through our Matlab plots.

Now let us consider the model exhibiting the Allee effect. It is important to note that this model has two equilibria, one unstable n_1^* and one stable n_2^* with $n_1^* < n_2^*$. By examining the differential equation $\frac{dn}{dt}$ we see that if a population is harvested too aggressively with $f \geq (\frac{1+a}{2})^2$ then as $t \to \infty$ the population will go extinct ie. $n(t) = 0$. This means we only want to catch fish at a rate $f < (\frac{1+a}{2})^2$.

Then given $f < (\frac{1+a}{2})^2$ to keep the population from going extinct, we can then determine the smallest possible population density that will not cause the population to go extinct. When $n_0 < n_1^*$ then as $t \to \infty$ the population density decreases to zero; however when $n_0 > n_1^*$ the population

increases to the stable equilibrium of n_2^* so the initial population size must be greater than the unstable equilibrium n_1^* before we begin harvesting, if not the population goes extinct.

In fishery management, it is important to determine the maximum rate at which fish can be harvested given a certain population so given a fish population with current population n_0 then we want to find the maximum harvest rate that does not kill off the population. If $\frac{dn}{dt} < 0$ then the population will go extinct. This means if $f > (n_0 - a)(1 - n_0)$ then as time increases $n(t)$ decreases to zero so the population moves towards extinction. If $f \leq (n_0 - a)(1 - n_0)$ then the population is sustainable.

6. Discussion & Conclusion

The economic and ecological dilemma of how to best catch fish plays a major role in fishery management. This research has given a method of finding the optimal harvest rate that maximizes yield given a fish population. Further studies include examining other harvesting strategies. The general case for a time dependent model would be any continuous function $f(t)$; however this would logistically be difficult to implement. Fisherman are not continually fishing and in the findustry the concern is annual yield, so the logical choice is to use a constant harvest rate over various time steps. We could extend our two step harvesting function to three, four, or finitely many steps and see whether this would further increase yield in populations below the stable equilibrium.

The simple logistic, skewed logistic, and Allee effect models are based on a simplified single stage fish population. In some species the growth rate is strongly affected by the age of an individual in the population. If this were the case, we could extend our models to a multistage model such as a two stage model that considers juvenille and adult fish. Furthermore the models used in this project use the intrinsic growth rate as the only growth parameter. Other population models incorporate life history strategies and use parameters such as birth, death, fecundity, reproductive, and maturation rates. These rates all affect the overall intrinsic growth rate of the population and could be implemented to construct models for specific species. This would be beneficial because fisherman are interested in harvesting species that are in high economic demand.

References

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[2] MathWorks. *MATLAB.* Natick, Massachusetts. 2011.

[3] Wolfram Research, Inc.. *Mathematica Edition: Version 6.0.* Champaign, Illinois. 2007.