# Efficient Algorithm for Estimating Amoebae Using Archimedian Tropical Varieties

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**Abstract**

Given any complex Laurent polynomial $f$, Amoeba$(f)$ is defined as the image of the complex roots of $f$ under the coordinate-wise log absolute value map. The Archimedean tropical variety has been proposed as a means of approximating Amoeba$(f)$, and there is now an explicit bound on the Hausdorff distance between the two sets. We illustrate a polynomial-time sub-algorithm that computes the connected components of ArchTrop$(f)$ for a given query point. We then extend this sub-algorithm to an exponential-time algorithm for approximating the nearest root to a given query point for a system of polynomials and finding all intersections of Archimedean tropical varieties for polynomials of a given system.

## 1 Introduction

### 1.1 The Program

I had no prior experience with algebraic geometry before this program, and I feel as if I have learned a good deal about this branch of mathematics, thanks to my research advisor. Dr. Rojas gave us two weeks of instruction and lecture before we began research on our project, meeting with my research partners and me to clarify concepts and discuss results as our work progressed.

Besides learning about good deal about algebraic geometry, I was able to develop my programming skills in working on this project. My research partners and I spent most of our time coding in Matlab, and I am happy I had the opportunity to learn this valuable skill.

## 1.2 Introduction to Algebraic Geometry

The traditional means of solving poynomial systems of equations, particuarily with the use of Grobner Bases, unfortunately has a lower complexity bound of exponential space. It is fortunate, however, that polyhedtral geometry can be used to approximate the norms of roots of polynomials, a relationship first discovered by Isaac Newton over the field of Puiseux series. Because algebraic varieties can be defined in a tropical setting, becoming unions of (n-1)-dimensional polyhedra in $\mathbb{R}^n$, we look at tropical geometry, specifically the Archimedean case, as a means of linking polyhedral geometry with algebraic geometry in order to esimate the norms of roots for multi-variate, n-dimensional, systems of equations.

## 1.3 Definitions

We use the abbreviations $[N] := \{1, ..., N\}$, $x := (x_1, ..., x_n)$, and let Conv($S$) denote the convex hull of a set $S$. Let us then define the function Log$|x|$ to be $(log|x_1|, ..., log|x_n|)$ and, for any $f \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, we define Amoeba($f$) to be $\{Log|x| \mid f(x) = 0,\ x \in (\mathbb{C}^*)^n\}$.Also, writing $f(x) = \sum_{i=1}^{t} C_i x^{ai}$ with $C_i \neq 0$ for all $i$, we define the support (or spectrum) of $f$ to be Supp$(f)$ := $\{a_i\}_{i \in t}$,the (ordinary) Newton polytope of $f$ to be Newt$(f)$ :=Conv(Supp$(f)$), and the Archimedean Newton polytope of f to be ArchNewt$(f)$ :=Conv$(\{(a_i, -\log |c_i|)\}_{i \in t})$. We also define the Archimedean tropical variety of $f$, denoted ArchTrop$(f)$, to be the set of all $v \in \mathbb{R}^n$ with $(v, -1)$ an outer normal of a positive-dimensional face of ArchNewt$(f)$.Finally, given any subsets $R$, $S \in R^n$, their Hausdorff distance, $\Delta(R, S)$, is defined to be the maximum of $\sup\limits_{\rho \in \mathbb{R}} \inf\limits_{\sigma \in S} |\rho - \sigma|$, and $\sup\limits_{\sigma \in S} \inf\limits_{\rho \in \mathbb{R}} |\rho - \sigma|$ where $|.|$ denotes the usual $L_2$ norm on $\mathbb{R}^n$.

## 1.4 Theorem

For any $f \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ with exactly $t$ monomial terms and Newt$(f)$ of dimension $k$, we have $t \geq k + 1$ and

$$\Delta(Amoeba(f), ArchTrop(f)) \leq (2t - 3)log(t - 1)$$

.

## 2 Algorithm

An algorithm that calculates the Archimedean tropical variety intersections lying on the connected component of the union of the Archimedean tropical varieties that surrounds a given query point for a system of $k$ $n$-dimensional polynomial equations is detailed as follows.

**Input:** For $k$ functions $f \in \mathbb{Q}[x_1, \ldots x_n]$, written as $f(x) = \sum_{i=1}^{t} C_i x^{ai}$ with $C_i \neq 0$ for all $i$ and the $a_i$ distinct, input each Supp($f$) concatenated with -log$|C_i|$ for the $k$ polynomials and a query point, $v \in \mathbb{Q}^n$.

**Output:** A matrix consisting of the intersections of the Archimedean tropical varieties of the $k$ polynomials in the connected component of $\mathbb{R}^n$ surrounding the query point $v$ and a matrix consisting of the distances from the query point, $v$, to the given ArchTrop intersections. **Description:**

1. Using linear programming, determine which vectors $\{a_i, -log|C_i|\}$ in Supp($f$), for each of the $k$ polynomials, cannot be written as a convex linear combination of the other vectors in the set. I.e., for any $\{a_k, -log|C_k|\}$, with $k \in [t]$, $\alpha_1(a_i, -log|C_i|) + \cdots + \alpha_n(a_t, -log|C_t|) \neq (a_k, -log|C_k|)$, the vector $(a_i, -\log|c_i)$ does not appear on the left hand side of the equation, and the real numbers $\alpha_i$ satisfy $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$. These vectors are the vertices defining the Archimedean Newton polytope, or ArchNewt, of the function.

2. Now determine which vertex of the ArchNewt($f$) yields the maximal inner product with the query point vector, $v$. Via the duality between ArchNewt($f$) and ArchTrop($f$) this vertex will correspond to the connected component of the ArchTrop(f) in which the given query point, $v$, will lie.

3. Using a second linear program, determine which vertices of ArchNewt($f$), when paired with the maximizing vertex, define edges of the ArchNewt($f$) for each of the $k$ polynomials. Consider the midpoints of all the line segments formed by a vertex other than the maximizing vertex being connected to the maximizing vertex. If a midpoint cannot be written as a convex linear combination of any vertices (other than, trivially, the two vertices it is halfway between), the edge from which it was derived defines an outer edge of the ArchNewt($f$). Thus, the outer edges of the hull emanating from the maximizing vertex are found.

4. In a third linear program, determine which edges defining ArchNewt($f$) for each of the $k$ polynomials are lower edges. Subtract any $\epsilon > 0$ from the last coordinate of the midpoint of each edge, and minimize a variable $s$ that is added to the last coordinate of the linear combination. If the minimum distance $s$ the midpoint's last coordinate may be raised while still remaining in the hull is less than $\epsilon$, the midpoint originally resided on an upper edge of the hull. If $s = \epsilon$, the midpoint originally resided on a lower edge of the hull. Thus, the lower edges of ArchNewt($f$) may be determined.

5. Now, for each of the $k$ polynomials, subtract the maximizing vertex from each vertex vector that helps define a lower edge of ArchNewt($f$) in order to obtain the vectors that define the lower edges of each ArchNewt($f$) of interest for the query point, $v$. Compute the inner product of each of these vectors with the vector $X = (x_1, x_2, \ldots, x_n, -1)$. Setting the product less than or equal to zero, the equations for the hyperplanes defining the connected component of the ArchTrop($f$) in which the query point lies are obtained.

6. Find all $k$-tuples from the $k$ sets of lower edge vectors in ArchNewt(f) for the $k$ polynomials. Check each $n$-dimensional $k$-tuple for linear independence, projecting the vectors "down a dimension" (i.e. ignoring the last coordinate).

7. Determine which of those linearly independent lower edge $k$-tuples have a Minkowski sum that defines a lower face of the Minkowski sum of the $k$ ArchNewt(f) they help define via a linear program similar to that in step 4.

8. Those $k$-tuples of edges whose Minkowski sums form lower faces on the Minkowski sum of the $k$ Archimedean Newton Polytopes define intersections of the $k$ Archimedean Tropical Varieties on the connected component of the union of the $k$ Archimedean Tropical Varieties that surrounds the query point. Taking the dot product of each edge in the $k$-tuple with $(x_1, x_2, \ldots, x_n, -1)$ yields a system of $k$ hyperplanes whose intersection is the intersection of the Archimedean Tropical Varieties we desire. From these intersecting hyperplanes, computing the intersections and (if $k = n$), the distance from $v$ to the intersection points, is trivial.

# 3 Preliminary Results

## 3.1 The POSSO Suite

We aim to test the efficiency and accuracy of our algorithm, as well as gain an indication of what the generalization of the Hausdorff bound in section 1.2 to arbitrary dimensions might look like. The POSSO suite provides an excellent opportunity to do this, offering numerous polynomial systems of varying sizes and orders that have been used in research in areas like chemistry and economics.

## 3.2 Results

For the Reimer 5 and Rose systems, most roots appear to lie less than 2 units from an intersection of the system's corresponding Archimedean tropical varieties. $\sup_{a \in \log|roots|} \inf_{b \in ArchTropIntersections} |a - b|$ initially appears to be relatively small, being 5.3872 for "Chemequs" and 0.5012 for "Caprasse." However, $\sup_{b \in ArchTropIntersections} \inf_{a \in \log|roots|} |a - b|$ seems to be much larger: 111.1693 for "Chemequs" and 5.8575 for "Caprasse."

# 4 Conclusion

## 4.1 Research

While we have completed our work for the summer, my research partner and I are planning to continue our work on the project, with the goal of publishing in a few months with Dr. Rojas. We will be completing testing of our algorithm using the Posso Suite, and we will be looking at the bounds on the Hausdorff distance between the known roots of a system of equations and the intersetions of the polynomials' Archimedean tropical varieties. We will additionally be looking at minimizing the time necessary for the algorithm to complete its computations.

## 4.2 The Program

I had a great experience at the REU program, and I look forward to continuing work with Sheridan and Dr. Rojas. While I am undecided as to what my future career plans are, this experience has helped shape my understanding of the academic math community and work within a research institution. I am thankful for the opportuntity to participate in this program, and I appreciate all the help and direction offered by my research mentors and partners.

# References


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