Algorithms for Determining the Topology of Positive Zero Sets

REU on Algorithmic Algebraic Geometry

Joseph Rennie

Texas A&M

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Outline

Algebraic Geometry-What

What is it?

- Varieties – Zero sets of systems of polynomials

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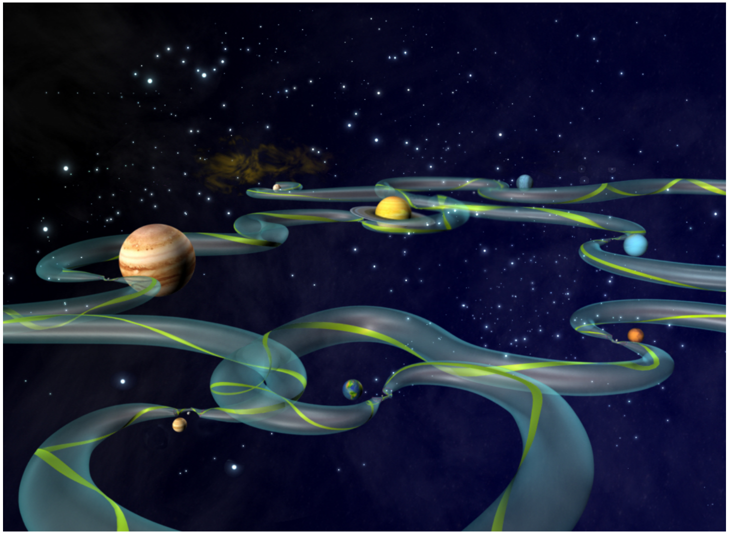
- Varieties – Zero sets of systems of polynomials
- Notation/Terminology Hell...but worth it!

Algebraic Geometry-Why

- Pure Mathematics
  - Nice Problems
- Connections to other areas of mathematics
  - Number Theory
  - Combinatorics
  - Statistics
- Applied Mathematics
  - Physics, Mathematical Biology, Automated Geometric Reasoning,...



(a) The "interplanetary superhighway"

Image can be found at www.jpl.nasa.gov/images/superhighway_square.jpg

Terms: at the gates

The **support** $\mathcal{A}$ of an n-variate t-nomial f, where

$$f(x_1, \ldots, x_n) = c_1 x^{a_1} + \cdots + c_t x^{a_t}$$

is given by $\mathcal{A} = \{a_1, \ldots, a_t\}$ where each $a_i \in \mathbb{R}^n$ and where $x^{a_i} = x_1^{a_{i1}} \ldots x_n^{a_{in}}$.

For example, let $f(x_1, x_2) = 42 + 42x_2^3 + 42x_1^3 + 42x_1x_2$ (a bivariate tetranomial) then $\mathcal{A} = \{(0,0),(0,3),(3,0),(1,1)\}$

A polynomial is said to be **honest** if its support does not lie in any (n-1)-plane.

Notation

- $Z_+(f)$ is the set of roots of f in the positive orthant $\mathbb{R}^n_+$.
- $Z_{\mathbb{R}}(f)$ is the set of real roots of f.

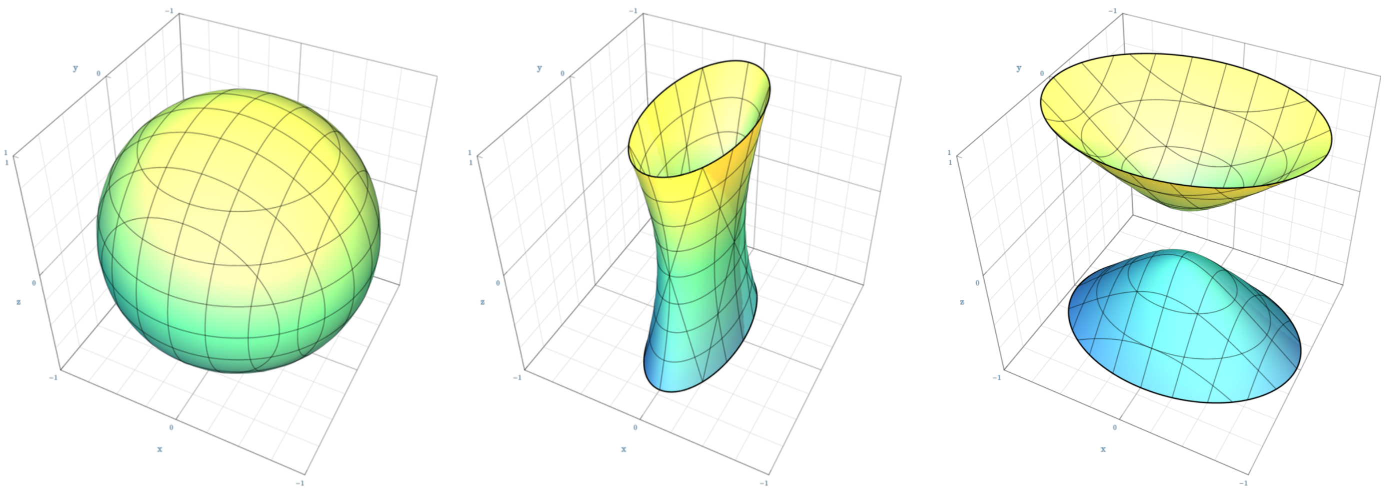
Conjecture

Given $f \in \mathbb{R}[x_1, \ldots, x_n]$ an honest $(n+2)$-nomial, $Z_+(f)$ has topology isotopic to a quadric hypersurface of the form

$$x_1^2 + \cdots + x_j^2 - (x_{j+1}^2 + \cdots + x_n^2) = \varepsilon$$

where j and the sign of ε are computable in polynomial time (for fixed n) from the support $\mathcal{A}$ and coefficients of f.

Quadric Hypersurfaces

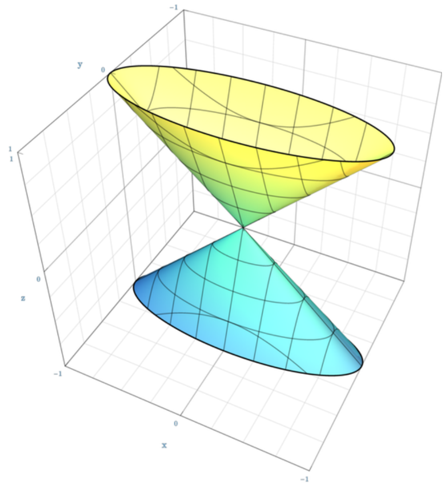


(b) $x_1^2 + x_2^2 + x_3^2 = 1$ (c) $x_1^2 + x_2^2 - x_3^2 = 1$ (d) $x_1^2 + x_2^2 - x_3^2 = -1$

Figure 1: Nondegenerate Quadric Hypersurfaces

Images courtesy of Wikipedia

Quadric Hypersurfaces



(A) $x_1^2 + x_2^2 - x_3^2 = 0$

Figure 2: Degenerate Quadric Hypersurface

Image courtesy of Wikipedia

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Discriminant Varieties

Given any $\mathcal{A} \in \mathbb{Z}^n$, we define the $\mathcal{A}$-discriminant variety, written $\nabla_{\mathcal{A}}$, to be the topological closure of

$$\{[c_1 : \cdots : c_T] \in \mathbb{P}_{\mathbb{C}}^{T-1} | c_1 x^{a_1} + \cdots + c_T x^{a_T} \text{ has a degenerate root in } (\mathbb{C}^*)^n\}$$

The real part of $\nabla_{\mathcal{A}}$ determines where in coefficient space the real zero set of a polynomial (with support $\mathcal{A}$) changes topology.

- Consequences of the conjecture
    - Tells us about the topology of positive zero sets of honest n-variate $(n+2)$-nomials of *arbitrary degree*
- Results
    - We currently have a bound on the number of connected components of n-variate $(n+2)$-nomials.

THANK YOU FOR YOUR ATTENTION!!!

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