ZEROS OF THE EISENSTEIN SERIES

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1. Introduction

It has been proved by Rankin and Swinnerton-Dyer[RS] that for the Eisenstein Series with $2k \geq 4$, the zeros of $E_{2k}(\tau)$ in the fundamental domain lie on the circle $|\tau| = 1$. This theorem has no parallel with respect to quasimodular forms. In fact, very little is known about the zeros of quasimodular forms. The Eisenstein Series of weight 2 is a quasimdoular form. It is known that the Eisenstein series has infinitely many zeros within the half-strip of the complex plane[BS]. However, apart from this fact not much is known about the location of these zeros. This paper will further investigate various properties of these zeros and the equivariant function $h(z)$.

2. Background

To begin, we would like to familiarize the reader with some terminology. The Fundamental Domain, denoted by D, is given by,

$$D = \{z \in \mathbb{H} : |z| \geq 1 \text{ and } -\frac{1}{2} \leq x \leq \frac{1}{2}\}.$$

The half-strip, denoted by G, is given by,

$$G = \{z \in \mathbb{H} : -\frac{1}{2} \leq x \leq \frac{1}{2}\}.$$

Let $SL_2(\mathbb{Z})$ be the set of matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$

We will now introduce the Eisenstein Series of weight $2k$ which has the Fourier expansion

$$E_{2k}(z) = 1 + \gamma_{2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z},$$

where

$$\gamma_{2k} = (-1)^k \frac{4k}{B_k},$$

B_k is the k-th Bernoulli number, and $\sigma_{2k-1}(n) = \sum_{a|n} a^{2k-1}$.

When $k \geq 2$, $E_{2k}(z)$ is a `modular form` for $SL_2(\mathbb{Z})$ which means that it is holomorphic on $\mathbb{H}$, including ∞, and it satisfies the relations

$$f(z) = (cz + d)^{-2k} f(\frac{az + b}{cz + d}) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

This is equivalent to

$$f(z + 1) = f(z) \quad \text{and} \quad f(\frac{-1}{z}) = z^{2k} f(z).$$

There are no non-zero modular forms of weight 2 for $SL_2(\mathbb{Z})$. When $k = 1$, the function $E_2(z)$ defined by its Fourier expansion

$$E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}$$

is not a modular form. Rather, it is called a quasimodular form, which satisfies the relation

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) - \frac{6}{\pi} ic(cz+d).$$

3. Zeros of $E_2(z)$

Basraoui and Sebbar [BS] showed that $E_2(z)$ has infinitely many zeros in G, none of which exist in D. However, very little is known about the actual location of these zeros. We used Mathematica to numerically solve the equation $E_2(z) = 0$ for $y \geq \varepsilon$ for various values of ε.

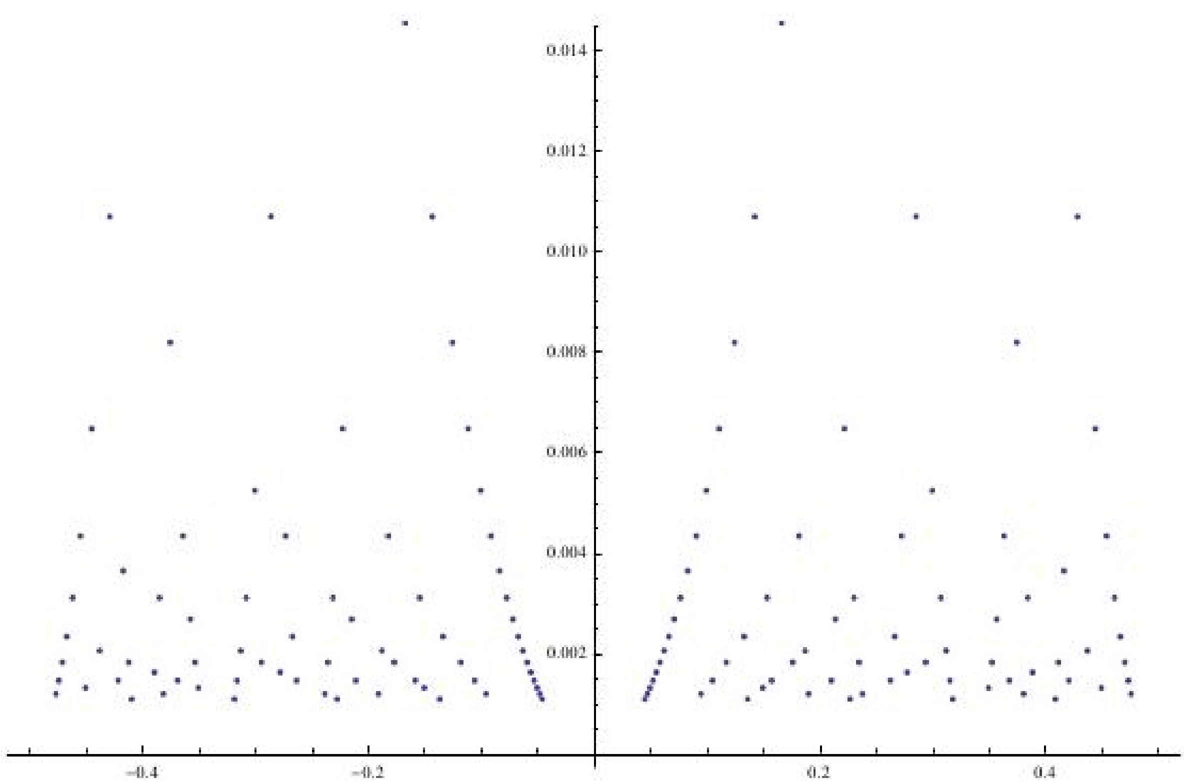

Figure 3.1. $E_2(z) = 0$ for $y > .001$.

As we were examining the data, we noticed that for $z = x + iy$ such that $E_2(z) = 0$, $\text{Re}(z)$ is very close to a rational number with a small denominator. Indeed, when we took our data points and applied a rational approximation out to the 4th decimal place, we found that all the rational numbers within G were represented to a certain limit that increases as ϵ gets closer to 0. We will display a small amount of our output for the reader to see. Note: Although we are showing both the x,y coordinates, we are only interested in the x-coordinate.

(-0.5, 0.13091903039678807) is equal to $-\frac{1}{2}$
(-0.3333258907443707, 0.0581819236539682) is very close approximation to $-\frac{1}{3}$
(-0.24999517436865332, 0.03272491502484815) is a very close approximation to $-\frac{1}{4}$
(-0.19999706592659725, 0.020942992285893466) is a very close approximation to $-\frac{1}{5}$
(-0.400001820482515, 0.020946451273604345) is a very close approximation to $-\frac{2}{5}$

And this pattern continues for all the zeros of $E_2(z)$ where $y > .001$. When we approximate these zeros, we generate a list of rational numbers. What you see below is just a small sample but is indiciative our results. Notice that every rational number (within $\mathbb{G}$) appears in the list out to a certain denominator. In this case, we stop our output at $\frac{3}{10}$, but be assured this pattern continues.

$$0, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{4}, -\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{2}{5},$$

$$-\frac{1}{6}, \frac{1}{6}, -\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8},$$

$$-\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, -\frac{3}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \ldots\ldots.$$

4. Properties of $h(z)$

We now introduce the equation for $h(z)$, which is defined by

$$h(z) = z + \frac{6}{i\pi E_2(z)}.$$

This function $h(z)$ is equivariant, which means that for $z \in \mathbb{H}$ and $\alpha \in SL_2(\mathbb{Z})$, then

$$h(\alpha z) = \alpha h(z). \tag{4.1}$$

See Sebbar-Sebbar [SS] for some properties of h. Also note that $h(z_0) = \infty$ is equivalent to $E_2(z_0) = 0$.

Proof of (4.1). Let $\alpha \in SL_2(\mathbb{Z}) : \alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in \mathbb{H}$

Consider $h(\alpha z) = \alpha z + \frac{6}{\pi i E_2(\alpha z)}$. By the transformation properties of $E_2(\alpha z)$ and of α we have,

$$h(\alpha z) = \frac{az+b}{cz+d} + \frac{6}{\pi i[(cz+d)^2 E_2(z) + \frac{6}{\pi i}c(cz+d)]}.$$

$$h(\alpha z) = \frac{1}{cz+d}\left[(az+b) + \frac{\frac{6}{\pi i E_2(z)}}{(cz+d) + \frac{6c}{\pi i E_2(z)}}\right]$$

$$h(\alpha z) = \frac{1}{cz+d}\left[\frac{(az+b)[(cz+d)+\frac{6c}{\pi i E_2(z)}]+\frac{6}{\pi i E_2(z)}}{(cz+d)+\frac{6c}{\pi i E_2(z)}}\right]$$

Recall, since $\alpha \in SL_2(\mathbb{Z})$, $\mathrm{ad}-\mathrm{bc}=1$.

$$h(\alpha z) = \frac{1}{cz+d}\left[\frac{(az+b)(cz+d)+(az+b)[\frac{6c}{\pi i E_2(z)}]+\frac{6}{\pi i E_2(z)}(ad-bc)}{(cz+d)+\frac{6c}{\pi i E_2(z)}}\right]$$

$$h(\alpha z) = \frac{1}{cz+d}\left[\frac{(az+b)(cz+d)+a(cz+d)\frac{6}{\pi i E_2(z)}}{(cz+d)+\frac{6c}{\pi i E_2(z)}}\right]$$

$$h(\alpha z) = \frac{(az+b)+a(\frac{6}{\pi i E_2(z)})}{(cz+d)+c(\frac{6}{\pi i E_2(z)})}$$

$$h(\alpha z) = \frac{a(z+\frac{6}{\pi i E_2(z)})+b}{c(z+\frac{6}{\pi i E_2(z)})+d} = \frac{a(h(z))+b}{c(h(z))+d} = \alpha h(z)$$

□

Next we state a variation of Lemma 3.4 of Balasubramanian-Gun [BG], who worked with $g(z) = 1/h(z)$.

Theorem 4.1. *If* $E_2(z_0) = 0$ *then* $h(\gamma z_0) = \frac{a}{c}$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. *Conversely, if* $h(\tau_0) = \frac{a}{c}$ *with coprime* a, c, *then* $E_2(\gamma^{-1}\tau_0) = 0$ *for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Proof. Consider the case when $E_2(z_0) = 0$ (so $h(z_0) = \infty$), and let $z = \gamma z_0$. Note that $\gamma\infty = \frac{a}{c}$. Then

$$h(\gamma z_0) = \gamma h(z_0) = \gamma\infty = \frac{a}{c}.$$

Conversely, suppose $h(\tau_0) = \frac{a}{c}$. Then

$$h(\gamma^{-1}\tau_0) = \gamma^{-1}h(\tau_0) = \gamma^{-1}\frac{a}{c} = \infty, \tag{4.2}$$

so $E_2(\gamma^{-1}\tau_0) = 0$. □

Since $h(z)$ is rational only when $E_2(z) = 0$, by graphing $\mathrm{Im}(h(z)) = 0$, all of the solutions to $E_2(z) = 0$ will be plotted along with some other values. The graphs of $\mathrm{Im}(h(z)) = 0$ are placed at the end of the paper, but the images will be discussed here.

The "almost-circular" shapes in Figures 4.1 and 4.2 can be shown to be nearly circlular. By applying all the elements of $SL_2(\mathbb{Z})$ to the curve that satisfies $\mathrm{Im}(h(z)) = 0$ in $\mathbb{D}$, our resulting images are the nearly circular shapes that we see below $\mathbb{D}$. Therefore, our curve in $\mathbb{D}$ is the generating curve for all the solution curves in $\mathbb{H}$. The curve generated by $\mathrm{Im}(h(z)) = 0$ in $\mathbb{D}$ can be bounded above and below by straight lines. By this fact, we know the curves below $\mathbb{D}$ are very close to circles. Because as $SL_2(\mathbb{Z})$ translates these 2 lines and our curve, the image is two perfect circles with the translated curve sitting in-between.

These graphs prompted us to ask the question: What would happen if we transformed the zeros of E_2 into $\mathbb{D}$? Recall, $E_2(z)$ has no zeros in $\mathbb{D}$. However, we can translate the coordinates of each of our zeros back into $\mathbb{D}$ by applying different elements of $SL_2(\mathbb{Z})$ to each individual point. Figure 4.4 shows a sample of some of our zeros of $E_2(z)$ which have been translated back into $\mathbb{D}$. When we show the curve of $\mathrm{Im}(h(z)) = 0$ which is in $\mathbb{D}$ along with our translated zeros, we have Figure 4.5. The fact that these lie on the same curve is a expected consequence of Theorem 4.1 and the fact that the curve of $\mathrm{Im}(h(z)) = 0$ in $\mathbb{D}$ is the generating curve for all values of curve of $\mathrm{Im}(h(z)) = 0$. Indeed, if we translated all of the zeros of $E_2(z)$ back into $\mathbb{D}$, the resulting image would be the same as Figure 4.3 (i.e. they would trace out the curve of $\mathrm{Im}(h(z)) = 0$ in $\mathbb{D}$).

Theorem 4.2 (M. Young, R. Wood)**.** *The real values of the function $h(z)$ which occur in the fundamental domain D occur only in the small strip $|y - 6/\pi| < .00028$.*

Proof. (Note: A more rigourous proof with actual bounds for the error terms is in progress)

Let $z \in D$,therefore, $\frac{\sqrt{3}}{2} \leq y \leq \infty$.

$$h(z) = (x + iy) + \frac{6}{\pi i E_2(x + iy)}$$

$$h(x + iy) = (x + iy) + \frac{6}{\pi i[1 - 24\sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi in(x+iy)}]}$$

$$h(x + iy) \approx (x + iy) + \frac{6}{\pi i}\frac{1}{1 - 24e^{2\pi ix}e^{-2\pi y} + \varepsilon}$$

where ε is a negligable error term.

Recall, the Taylor Series for $\frac{1}{1-u} = 1 + u + u^2 + u^3 + ...$ Therefore if we let

$$u = 24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon$$

we find that

$$h(x + iy) \approx (x + iy) + \frac{6}{\pi i}[1 + u + u^2 + u^3 + ...].$$

$$h(x + iy) \approx (x + iy) + \frac{6}{\pi i}\left[1 + (24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon) + (24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon)^2 + ...\right]$$

$\varepsilon_0 = \left[(24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon)^2 + (24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon)^3 + (24(1)e^{2\pi ix}e^{-2\pi y} + \varepsilon)^4 + ...\right]$ is also another negligable error term for our purposes.

$$h(x + iy) \approx (x + iy) + \frac{6}{\pi i}(1 + (24(1)e^{2\pi ix}e^{-2\pi y}))$$

$$h(x + iy) \approx x + iy + \frac{6}{\pi i} + \frac{24 * 6}{\pi i}e^{-2\pi y}[\cos 2\pi x + i \sin 2\pi x] + ...[\text{small error}]$$

$$h(x + iy) \approx \left[x + \frac{24 * 6}{\pi}\sin 2\pi x e^{-2\pi y}\right] + i\left[y - \frac{6}{\pi} - \frac{24 * 6}{\pi}[\cos 2\pi x e^{-2\pi y}]\right]$$

If $y > \frac{6}{\pi} + \frac{24*6}{\pi}e^{-2\pi y} + ...$, then $h(x + iy)$ is not real.

If $y < \frac{6}{\pi} - \frac{24*6}{\pi}e^{-2\pi y} + ...$, then $h(x + iy)$ is not real.

For $h(x + iy)$ to be real we need,

$$\frac{6}{\pi} - \frac{24 * 6}{\pi}e^{-2\pi y} \leq y \leq \frac{6}{\pi} + \frac{24 * 6}{\pi}e^{-2\pi y}$$

Let $y = \frac{6}{\pi} + \delta$ where delta is very small.

$$\frac{6}{\pi} - \frac{24*6}{\pi}e^{-2\pi y} \leq \frac{6}{\pi} + \delta \leq \frac{6}{\pi} + \frac{24*6}{\pi}e^{-2\pi y}$$

$$-\frac{24*6}{\pi}e^{-2\pi(\frac{6}{\pi}+\delta)} \leq \delta \leq \frac{24*6}{\pi}e^{-2\pi(\frac{6}{\pi}+\delta)}$$

$$-\frac{24*6}{\pi}e^{-12}e^{-2\pi\delta} \leq \delta \leq \frac{24*6}{\pi}e^{-12}e^{-2\pi\delta}$$

$$-\frac{24*6}{\pi}e^{-12} \leq e^{2\pi\delta}\delta \leq \frac{24*6}{\pi}e^{-12}$$

$e^{2\pi\delta} \approx 1$ since delta is very small and $e^0 = 1$

$$-\frac{24*6}{\pi}e^{-12} \leq \delta \leq \frac{24*6}{\pi}e^{-12}$$

$$|\delta| \leq \frac{24*6}{\pi}e^{-12} \approx .00028$$

$$|y - \frac{6}{\pi}| \leq \frac{24*6}{\pi}e^{-12} \approx .00028$$

□

Therefore, by Theorem 4.1 and Theorem 4.2, we can conclude that all of the zeros of $E_2(z)$ can be translated back into $\mathbb{D}$ and lie on the curve bounded above and below by

$$|y - \frac{6}{\pi}| \leq \frac{24*6}{\pi}e^{-12} \approx .00028\ .$$

References

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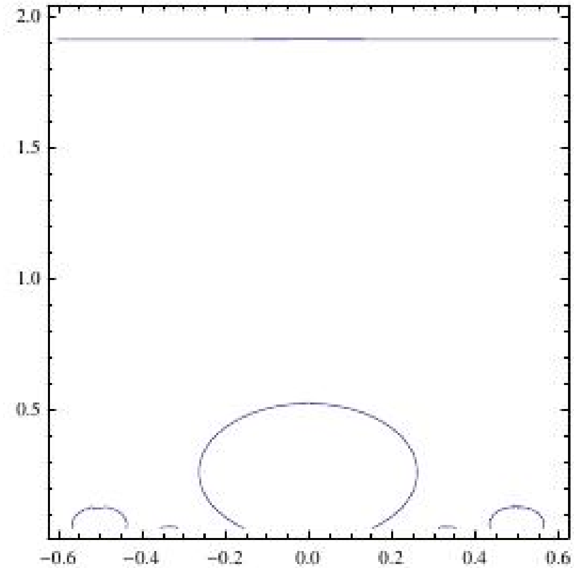

FIGURE 4.1. Graph of $Im(h(z)) = 0$

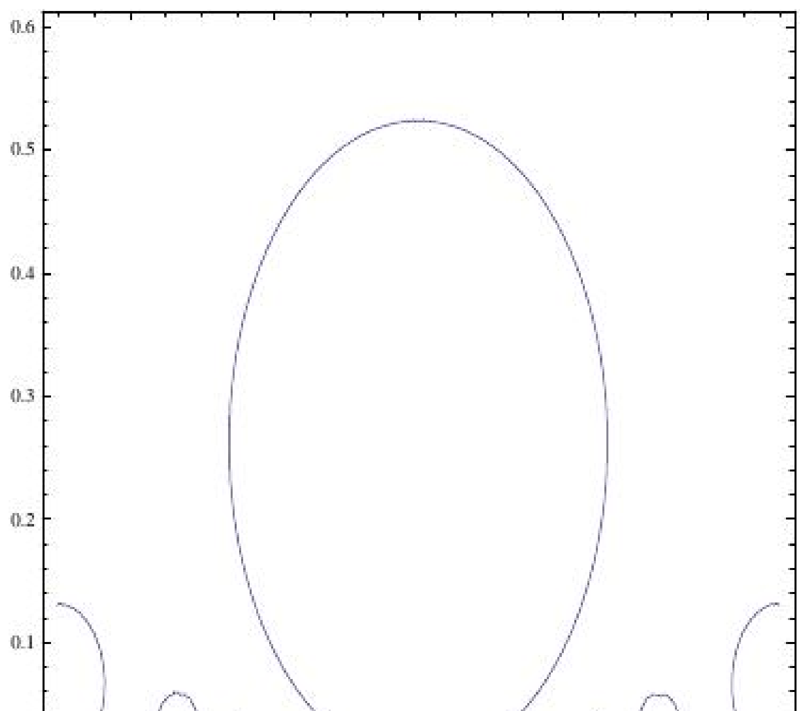

FIGURE 4.2. Zoomed in graph of $Im(h(z)) = 0$ below $\mathbb{D}$

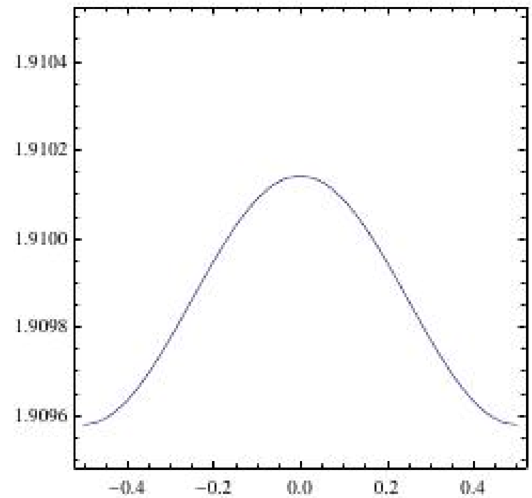

FIGURE 4.3. Zoomed in graph of $Im(h(z)) = 0$ in $\mathbb{D}$

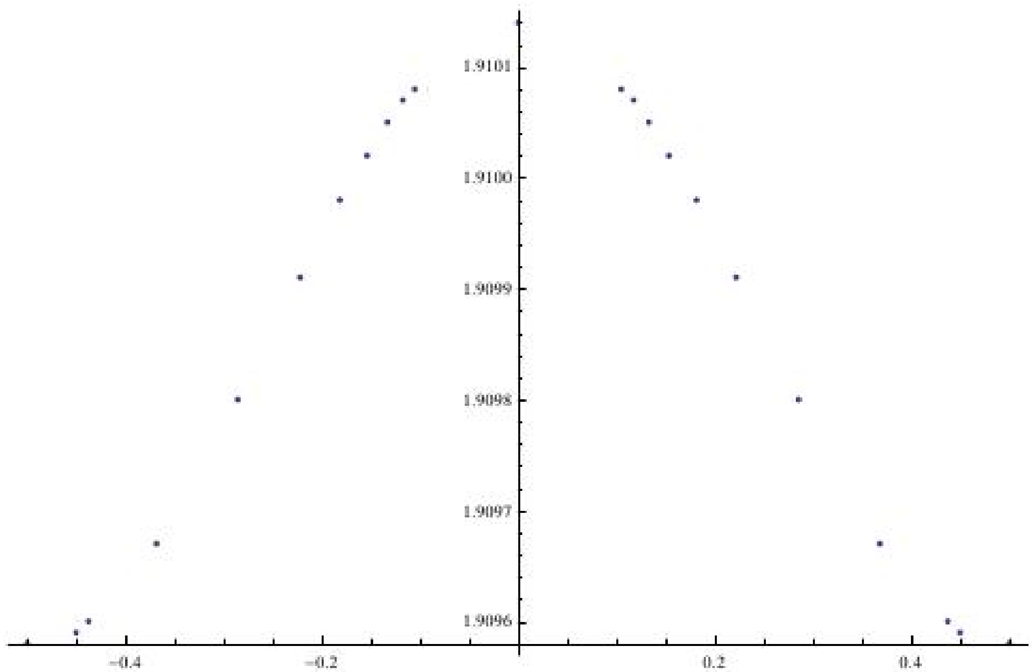

FIGURE 4.4. $SL_2(\mathbb{Z})$ Translated Zeros of $E_2(z) = 0$ into $\mathbb{D}$

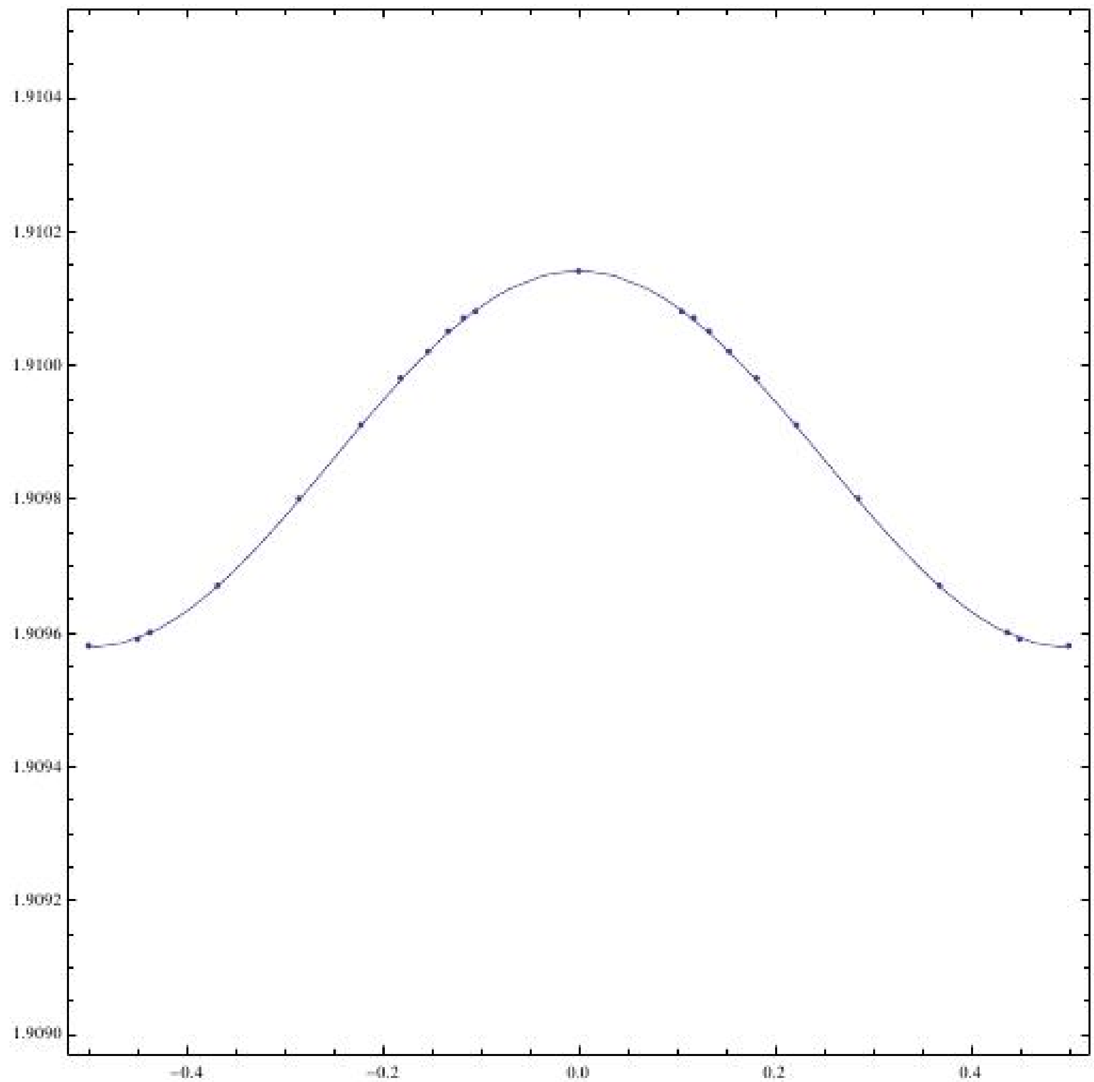

Figure 4.5. Plot of Translated Zeros of $E_2(z) = 0$ and $Im(h(z)) = 0$ in $\mathbb{D}$