SOME ARITHMETIC PROBLEMS RELATED TO CLASS GROUP L–FUNCTIONS

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Abstract. We prove that for each fundamental discriminant $-D < 0$ there exists at least one ideal class group character χ of $\mathbb{Q}(\sqrt{-D})$ such that the L–function $L(\chi, s)$ is nonvanishing at $s = \frac{1}{2}$. In addition, assuming that $L(\chi_0, \frac{1}{2}) \leq 0$ where χ_0 is the trivial character, we prove that the class number $h(-D)$ satisfies the effective lower bound

$$h(-D) \geq 0.1265 \cdot \varepsilon D^{\frac{1}{4}} \log(D)$$

for each fundamental discriminant $-D < 0$ with $D \geq (8\pi/e^{\gamma})^{(\frac{1}{2}-\varepsilon)^{-1}}$ where $0 < \varepsilon < 1/2$ is arbitrary and fixed (here γ is Euler's constant).

1. Introduction and statement of results

It is well known that the L–function $L(\chi, s)$ of an ideal class group character χ of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ can be expressed in terms of values of the real-analytic Eisenstein series for $SL_2(\mathbb{Z})$ at Heegner points. In this paper we will exploit this relationship to study two related arithmetic problems. First we show that for each fundamental discriminant $-D < 0$, there exists at least one χ such that the central value $L(\chi, \frac{1}{2}) \neq 0$. Next, by building on ideas of Iwaniec and Sarnak [IS] and Kowalski and Iwaniec [IK], we show that if $L(\chi_0, \frac{1}{2}) \leq 0$ where χ_0 is the trivial character, then the class number $h(-D)$ of K satisfies the effective lower bound

$$h(-D) \geq 0.1265 \cdot \varepsilon D^{\frac{1}{4}} \log(D)$$

for each fundamental discriminant $-D < 0$ with $D \geq (8\pi/e^{\gamma})^{(\frac{1}{2}-\varepsilon)^{-1}}$ where $0 < \varepsilon < 1/2$ is arbitrary and fixed (here γ is Euler's constant).

In order to discuss these results in more detail we fix the following notation. Let $-D < 0$ be a fundamental discriminant, $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, $\mathcal{O}_D$ be the ring of integers, ω be the number of units in $\mathcal{O}_D$, $Cl(\mathcal{O}_D)$ be the ideal class group of K, $h(-D)$ be the class number, and $\widehat{Cl(\mathcal{O}_D)}$ be the group of characters of $Cl(\mathcal{O}_D)$. Given $\chi \in \widehat{Cl(\mathcal{O}_D)}$, the class group L-function is defined by

$$L(\chi, s) = \sum_{\mathcal{C} \in Cl(\mathcal{O}_D)} \chi(\mathcal{C}) \zeta_{\mathcal{C}}(s),$$

where

$$\zeta_{\mathcal{C}}(s) = \sum_{\substack{0 \neq \mathfrak{a} \in \mathcal{C} \\ \mathfrak{a} \text{ integral}}} N(\mathfrak{a})^{-s}, \qquad \mathfrak{Re}(s) > 1$$

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and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$. It is known that if χ is nontrivial, then $L(\chi, s)$ extends to an entire function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s),$$

where

$$\Lambda(s) := (2\pi)^{-s}\Gamma(s)D^{s/2}L(\chi, s).$$

The central value is $L(\chi, \frac{1}{2})$.

The nonvanishing of central values of automorphic L-functions is a problem of great importance in number theory. While it is difficult to determine whether an individual L–function is nonvanishing, progress can often be made by studying L-functions in families. The class group L–functions provide an interesting example of such a family (see [DFI], [FI], [B]). The nonvanishing of their central values was studied by Blomer [B], who used deep techniques in analytic number theory to prove that as $D \to \infty$,

$$\frac{\#\{\chi \in \widehat{Cl(\mathcal{O}_D)} : L\left(\chi, \frac{1}{2}\right) \neq 0\}}{h(-D)} \geq c \prod_{p|D} \left(1 - \frac{1}{p}\right) \tag{1}$$

for some explicit $c > 0$. This result is ineffective in the sense that one does not know how large D must be for (1) to hold due to an application of Siegel's theorem in the proof.

We will show that there is always at least one $\chi \in \widehat{Cl(\mathcal{O}_D)}$ such that $L(\chi, \frac{1}{2}) \neq 0$.

Theorem 1.1. *For each fundamental discriminant* $-D < 0$ *there exists at least one* $\chi \in \widehat{Cl(\mathcal{O}_D)}$ *such that* $L\left(\chi, \frac{1}{2}\right) \neq 0$.

It is expected that for each $-D < 0$ one has $L\left(\chi, \frac{1}{2}\right) \neq 0$ for *all* $\chi \in \widehat{CL(\mathcal{O}_D)}$. We have calculated the following table of L-values for small D with prime class number $h(-D)$ using the identity (2).

Table 1. L-function Values for Small D.

$-D$	$h(-D)$	L-function Values at $s = \frac{1}{2}$
−3	1	−0.702237
−4	1	−0.975066
−7	1	−1.67442
−8	1	−1.60701
−11	1	−1.44805
−15	2	−2.69732, 0.111442
−19	1	−1.17474
−20	2	−2.45292, 0.154738
−23	3	−3.5857, 0.174036, 0.174036
−24	2	−2.29555, 0.179696
⋮	⋮	⋮
−47	5	−4.82435, 0.359728, 0.247743, 0.247743, 0.359728
⋮	⋮	⋮
−71	7	−5.99259, 0.521411, 0.417899, 0.252331, 0.252331, 0.417899, 0.521411
⋮	⋮	⋮

Our proof of Theorem 1.1 is inspired by the notion of "quantification in the cusp" discussed by Michel and Venkatesh in [MV]. To study the nonvanishing of $L(\chi, \frac{1}{2})$ we use the following exact formula for the first moment,

$$\frac{1}{h(-D)} \sum_{\chi \in \widehat{Cl(\mathcal{O}_D)}} L(\chi, \tfrac{1}{2}) = \frac{2}{w} \left(\frac{\sqrt{D}}{2} \right)^{-\frac{1}{2}} f(z_{\mathcal{O}_D}, \tfrac{1}{2}),$$

where $f(z, \frac{1}{2})$ is essentially the central derivative of the real-analytic Eisenstein series for $SL_2(\mathbb{Z})$ and $z_{\mathcal{O}_D}$ is the Heegner point corresponding to the trivial ideal class (see Propositions 2.1 and 2.2). The Heegner point $z_{\mathcal{O}_D}$ lives high in the cusp of $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Since the Fourier expansion of $f(z, \frac{1}{2})$ is dominated by its constant term, which depends only on the imaginary part of z, we are able to prove that $f(z_{\mathcal{O}_D}, \frac{1}{2}) \neq 0$ for all D.

Another problem of great importance in number theory is that of finding effective lower bounds for the class number $h(-D)$. Very strong effective lower bounds which are conditional on the location of the zeros of the quadratic Dirichlet L–function $L(\chi_D, s)$ have been known for many years. For example, in 1918, Hecke and Landau proved that if $L(\chi_D, s)$ does not vanish in the region $s > 1 - a/\log(D)$ then

$$h(-D) > b\frac{D^{\frac{1}{2}}}{\log(D)}$$

where a and b are effective, positive constants (see [IK, Proposition 22.2]). However, it is natural to ask if strong effective lower bounds can be obtained without assuming anything about the location of the zeros. This question was discussed by Iwaniec and Sarnak [IS, section 5] in the context of nonnegativity of central values of automorphic L–functions. Namely, Iwaniec and Sarnak remarked that if one knew that $L(\chi_D, \frac{1}{2}) \geq 0$, then one could "eliminate in part the Landau-Siegel lacuna" discussed in [IS, section 2].[1] This idea was revisited in Iwaniec and Kowalski [IK, section 22.3], where they explained how to use the condition $L(\chi_D, \frac{1}{2}) \geq 0$ to establish an effective lower bound of the form

$$h(-D) \gg D^{\frac{1}{4}} \log(D).$$

Using methods similar to those in the proof of Theorem 1.1, we will elaborate on the argument of Iwaniec and Kowalski and make this lower bound completely explicit.

Theorem 1.2. *Let $-D < 0$ be a fundamental discriminant with $D \geq (8\pi/e^{\gamma})^{(\frac{1}{2}-\varepsilon)^{-1}}$ where $0 < \varepsilon < 1/2$ is arbitrary and fixed (here γ is Euler's constant). Assume that $L(\chi_D, \frac{1}{2}) \geq 0$. Then*

$$h(-D) \geq 0.1265 \cdot \varepsilon D^{\frac{1}{4}} \log(D).$$

It is not difficult to show that GRH implies $L(\chi_D, \frac{1}{2}) \geq 0$. As remarked by Iwaniec and Kowalski [IK, section 22.3], this result "may conceivably be established sometime without recourse to the GRH". In fact, it should be emphasized that there are many examples of automorphic L–functions which are known unconditionally to have nonnegative central values (see [IS, section 5]). This is striking considering the difficulty of proving such a result in the "simplest" case of a quadratic Dirichlet L–function.

[1] The condition $L(\chi_0, \frac{1}{2}) \leq 0$ is equivalent to $L(\chi_D, \frac{1}{2}) \geq 0$ since $L(\chi_0, s) = \zeta(s)L(\chi_D, s)$.

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2. Averaging L-functions

Recall that each ideal class $\mathcal{C} \in Cl(\mathcal{O}_D)$ contains a reduced, primitive integral ideal

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{b+\sqrt{-D}}{2}\right)$$

with $a = N(\mathfrak{a})$. Moreover, the point

$$z_{\mathfrak{a}} = \frac{b+\sqrt{-D}}{2a}$$

lies in the standard fundamental domain for $\Gamma = SL_2(\mathbb{Z})$. Following convention, we call these points *Heegner points*. Next define the Eisenstein series

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathfrak{Im}(\gamma z)^s, \qquad \mathfrak{Re}(s) > 1$$

where

$$\Gamma_\infty = \left\{ \left(\begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) \middle| \, n \in \mathbb{Z} \right\}.$$

We connect the central L-values, Eisenstein series, and Heegner points with the following

Proposition 2.1. *We have*

$$\frac{1}{h(-D)} \sum_{\chi \in \widehat{Cl(\mathcal{O}_D)}} L(\chi, s) = \frac{2}{w}\zeta(2s)\left(\frac{\sqrt{D}}{2}\right)^{-s} E(z_{\mathcal{O}_D}, s).$$

Proof. Recall the following classical formula due to Hecke,

$$\zeta_{[\mathfrak{a}]}(s) = \frac{2}{w}\zeta(2s)\left(\frac{\sqrt{D}}{2}\right)^{-s} E(z_{\mathfrak{a}}, s).$$

Then

$$L(\chi,s) = \frac{2}{w}\zeta(2s)\left(\frac{\sqrt{D}}{2}\right)^{-s} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_D)} \chi(\mathfrak{a}) E(z_{\mathfrak{a}}, s), \tag{2}$$

so we have

$$\sum_{\chi \in \widehat{Cl(\mathcal{O}_D)}} L(\chi,s) = \frac{2}{w}\zeta(2s)\left(\frac{\sqrt{D}}{2}\right)^{-s} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_D)} E(z_{\mathfrak{a}}, s) \sum_{\chi \in \widehat{Cl(\mathcal{O}_D)}} \chi(\mathfrak{a}).$$

By the orthogonality relations,

$$\sum_{\chi \in \widehat{Cl(\mathcal{O}_D)}} \chi(\mathfrak{a}) = \begin{cases} h(-D), & [\mathfrak{a}] = [\mathcal{O}_D] \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\frac{1}{h(-D)}\sum_{\chi\in\widehat{Cl(\mathcal{O}_D)}} L(\chi,s)=\frac{2}{w}\zeta(2s)\left(\frac{\sqrt{D}}{2}\right)^{-s}E(z_{\mathcal{O}_D},s).$$

□

Proposition 2.2. *Let*

$$f(z,s):=\zeta(2s)E(z,s).$$

Then for $z=x+iy$ we have

$$\begin{aligned}f(z,s)&=\sqrt{y}\left(\log(y)-\log(4\pi e^{-\gamma})\right)\\&\quad+4\sqrt{y}\sum_{n=1}^{\infty}\tau_{s-\frac12}(n)K_{s-\frac12}(2\pi ny)\cos(2\pi nx)+O(s-\tfrac12),\end{aligned}$$

where γ is Euler's constant,

$$\tau_s(n)=\sum_{ab=n}\left(\frac{a}{b}\right)^s$$

and $K_s(t)$ is the K-Bessel function.

Proof. Recall the Fourier expansion (see [IK, eq. (22.46)])

$$\begin{aligned}f(z,s)&=y^s\zeta(2s)+\sqrt{\pi}\frac{\Gamma(s-\frac12)\zeta(2s-1)}{\Gamma(s)}y^{1-s}\\&\quad+\frac{4\pi^s}{\Gamma(s)}\sqrt{y}\sum_{n=1}^{\infty}\tau_{s-\frac12}(n)K_{s-\frac12}(2\pi ny)\cos(2\pi nx).\end{aligned}$$

We have

$$\zeta(2s)=\frac{1}{2(s-\frac12)}+\gamma+O(s-\tfrac12)$$

and

$$y^s=y^{\frac12}(1+\log y(s-\tfrac12)+O(s-\tfrac12)),$$

thus

$$y^s\zeta(2s)=\frac{y^{\frac12}}{2(s-\frac12)}+\gamma y^{\frac12}+\frac12 y^{\frac12}\log y+O(s-\tfrac12).$$

Also recall the functional equation

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}.$$

Making the change of variables $s\to 2s-1$ in this equation yields

$$\pi^{\frac12-s}\Gamma(s-\tfrac12)\zeta(2s-1)=\zeta(2-2s)\Gamma(1-s)\pi^{-(1-s)},$$

and thus

$$\sqrt{\pi}\frac{\Gamma(s-\frac12)\zeta(2s-1)y^{1-s}}{\Gamma(s)}=\frac{\pi^{2s-1}\Gamma(1-s)y^{1-s}}{\Gamma(s)}\zeta(2-2s).$$

We want to calculate the Taylor expansion of

$$\frac{\pi^{2s-1}\Gamma(1-s)y^{1-s}}{\Gamma(s)}\zeta(2-2s)$$

at $s = \frac{1}{2}$. We have

$$\zeta(2-2s) = \frac{1}{2(s-\frac{1}{2})} + \gamma + O(s-1),$$

and

$$\frac{\pi^{2s-1}\Gamma(1-s)y^{1-s}}{\Gamma(s)} = y^{\frac{1}{2}} + \alpha(s-\tfrac{1}{2}) + O(s-\tfrac{1}{2})^2,$$

where

$$\alpha := \frac{d}{ds}\left(\frac{\pi^{2s-1}\Gamma(1-s)y^{1-s}}{\Gamma(s)}\right)\Bigg|_{s=\frac{1}{2}}.$$

A straightforward calculation shows

$$\alpha = y^{\frac{1}{2}}(2\log\pi - \log y + 2\gamma + 2\log 4).$$

Putting things together, we get

$$\frac{\pi^{2s-1}\Gamma(1-s)y^{1-s}}{\Gamma(s)}\zeta(2-2s) = \frac{-y^{\frac{1}{2}}}{2(s-\frac{1}{2})} + \gamma y^{\frac{1}{2}} + (-\tfrac{1}{2})y^{\frac{1}{2}}(2\log\pi - \log y + 2\gamma + 2\log 4),$$

which after simplification gives

$$f(z,s) = \sqrt{y}(\log y - \log(4\pi e^{-\gamma})) + 4\sqrt{y}\sum_{n=1}^{\infty}\tau_{s-\frac{1}{2}}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx) + O(s-\tfrac{1}{2}).$$

□

3. Proof of Theorem 1.1

By combining Propositions 2.1 and 2.2, we obtain the identity

$$\frac{1}{h(-D)}\sum_{\chi\in\widehat{Cl(\mathcal{O}_D)}} L(\chi,\tfrac{1}{2}) = \frac{2}{w}\left(\frac{\sqrt{D}}{2}\right)^{-\frac{1}{2}} f(z_{\mathcal{O}_D},\tfrac{1}{2}),$$

where $z_{\mathcal{O}_D} = \frac{b+\sqrt{D}}{2}$, $x = \frac{b}{2}$, $y = \frac{\sqrt{D}}{2}$ and

$$f(z_{\mathcal{O}_D},\tfrac{1}{2}) = \sqrt{y}\big(\log(y) - \log(4\pi e^{-\gamma})\big) + 4\sqrt{y}\sum_{n=1}^{\infty}\tau_0(n)K_0(2\pi ny)\cos(2\pi nx).$$

It suffices to show $f(z_{\mathcal{O}_D},\frac{1}{2}) \neq 0$ for all D. We assume that $f(z_{\mathcal{O}_D},\frac{1}{2}) = 0$ for some D and obtain a contradiction. From [GR, (8.451.6)] we have that

$$K_0(t) = \sqrt{\frac{\pi}{2t}}e^{-t}\left(1 + \frac{\theta}{2t}\right)$$

for $t > 0$ where $|\theta| \leq \frac{1}{4}$. This, along with $y \geq \frac{\sqrt{3}}{2}$ allows us to bound the K-Bessel function term as

$$|K_0(2\pi ny)| \leq \left(1 + \frac{1}{8\sqrt{3}\pi}\right)\sqrt{\frac{1}{4ny}}e^{-2\pi ny}.$$

Using the elementary bound $\tau_0(n) \leq 2\sqrt{n}$, we bound the infinite sum in the Fourier expansion by

$$\begin{aligned}\left|4\sqrt{y}\sum_{n=1}^{\infty}\tau_0(n)K_0(2\pi ny)\cos(2\pi nx)\right| &\leq 4\sum_{n=1}^{\infty}|2\sqrt{ny}K_0(2\pi ny)| \\ &\leq \left(4+\frac{1}{2\sqrt{3}\pi}\right)\sum_{n=1}^{\infty}e^{-2\pi ny} \\ &= \left(4+\frac{1}{2\sqrt{3}\pi}\right)\cdot\frac{e^{-2\pi y}}{1-e^{-2\pi y}} \leq \left(4+\frac{1}{2\sqrt{3}\pi}\right)\cdot\frac{e^{-\pi\sqrt{3}}}{1-e^{-\pi\sqrt{3}}} \\ &< 0.018.\end{aligned}$$

Since $f(z_{\mathcal{O}_D}, \frac{1}{2}) = 0$, we have

$$|\sqrt{y}\left(\log(y) - \log(4\pi e^{-\gamma})\right)| = \left|4\sqrt{y}\sum_{n=1}^{\infty}\tau_0(n)K_0(2\pi ny)\cos(2\pi nx)\right| < .018.$$

Therefore

$$|\log(y) - \log(4\pi e^{-\gamma})| \leq \frac{2}{\sqrt{3}}|\sqrt{y}(\log(y) - \log(4\pi e^{-\gamma}))| < .021.$$

This implies

$$4\pi e^{-\gamma-.021} < y < 4\pi e^{-\gamma+.021},$$

so that $6.90 < y < 7.21$. But using this bound, we can improve the bound on the infinite sum to

$$\left|4\sqrt{y}\sum_{n=1}^{\infty}\tau_0(n)K_0(2\pi ny)\cos(2\pi nx)\right| \leq \left(4+\frac{1}{2\sqrt{3}\pi}\right)\frac{e^{-\pi\cdot 13.8}}{1-e^{-\pi\cdot 13.8}} < 6.08\cdot 10^{-19}.$$

Repeating the previous argument with this much sharper bound, we find that

$$4\pi e^{-\gamma-6.08\cdot 10^{-19}} < y < 4\pi e^{-\gamma+6.08\cdot 10^{-19}}$$

and

$$199.12076 < 4y^2 < 199.12077.$$

But since $y = \frac{\sqrt{D}}{2}$, we have $199 < D < 200$, so that $D \notin \mathbb{Z}$, a contradiction. Thus $f(z_{\mathcal{O}_D}, \frac{1}{2}) \neq 0$ for all D. □

4. Proof of Theorem 1.2

By (2) we have

$$L(\chi, \tfrac{1}{2}) = \frac{2}{w}\left(\frac{\sqrt{D}}{2}\right)^{-\frac{1}{2}}\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)}\chi(\mathfrak{a})f(z_{\mathfrak{a}}, \tfrac{1}{2}),$$

where $z_{\mathfrak{a}} = \frac{b+\sqrt{D}}{2N(\mathfrak{a})}$, $x = \frac{b}{2N(\mathfrak{a})}$, $y = \frac{\sqrt{D}}{2N(\mathfrak{a})}$ and

$$f(z_{\mathfrak{a}}, \tfrac{1}{2}) = \sqrt{y}(\log y - \log(4\pi e^{-\gamma})) + 4\sqrt{y}\sum_{n=1}^{\infty}\tau_0(n)K_0(2\pi ny)\cos(2\pi nx).$$

Therefore, a calculation yields

(3)
$$L(\chi,\tfrac{1}{2}) = \frac{2}{w}\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)} \frac{\chi(\mathfrak{a})}{\sqrt{N(\mathfrak{a})}}\left\{\log\left(\frac{\alpha\sqrt{D}}{N(\mathfrak{a})}\right) + 4\sum_{n=1}^{\infty}\tau_0(n)K_0\left(\frac{\pi n\sqrt{D}}{N(\mathfrak{a})}\right)\cos\left(\frac{\pi n b}{N(\mathfrak{a})}\right)\right\},$$

where $\alpha := e^{\gamma}/8\pi$.

Let $\chi = \chi_0$ be the trivial character. Then $L(\chi_0,\frac{1}{2}) = \zeta(\frac{1}{2})L(\chi_D,\frac{1}{2})$, where $L(\chi_D,s)$ is the Dirichlet L–function associated to the Kronecker symbol χ_D. Assume that $L(\chi_D,\frac{1}{2}) \geq 0$, so that $L(\chi_0,\frac{1}{2}) \leq 0$. Then by (3) we obtain

$$\frac{2}{\omega}\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)} \frac{1}{\sqrt{N(\mathfrak{a})}}\log\left(\alpha\frac{\sqrt{D}}{N(\mathfrak{a})}\right) \leq |E|, \tag{4}$$

where

$$E := \frac{2}{\omega}\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)} \frac{4}{\sqrt{N(\mathfrak{a})}}\sum_{n=1}^{\infty}\tau_0(n)K_0\left(\frac{\pi n\sqrt{D}}{N(\mathfrak{a})}\right)\cos\left(\frac{\pi n b}{N(\mathfrak{a})}\right).$$

Assume now that $\log(\alpha\sqrt{D}) \geq \varepsilon\log(D)$ for some arbitrary, fixed $0 < \varepsilon < 1/2$ (in particular, under this assumption $\omega = 2$). Split the sum on the left hand side of (4) as $S_1 + S_2$, where

$$S_1 := \sum_{1\leq N(\mathfrak{a})\leq\alpha\sqrt{D}} \frac{1}{\sqrt{N(\mathfrak{a})}}\log\left(\alpha\frac{\sqrt{D}}{N(\mathfrak{a})}\right),$$
$$S_2 := \sum_{\alpha\sqrt{D}<N(\mathfrak{a})\leq\sqrt{\frac{D}{3}}} \frac{1}{\sqrt{N(\mathfrak{a})}}\log\left(\alpha\frac{\sqrt{D}}{N(\mathfrak{a})}\right).$$

Then each summand in S_1 is nonnegative and we have

$$S_1 \leq |E| + |S_2|.$$

Using $N(\mathfrak{a}) \leq \sqrt{D/3}$, we argue as in the proof of Theorem 1.1 to obtain

$$\begin{aligned}
|E| &\leq \left(4 + \frac{1}{2\sqrt{3}\pi}\right)\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)}\sum_{n=1}^{\infty}\frac{2\sqrt{n}}{\sqrt{N(\mathfrak{a})}}\left(\frac{2n\sqrt{D}}{N(\mathfrak{a})}\right)^{-\frac{1}{2}}\exp\left(-\frac{\pi n\sqrt{D}}{N(\mathfrak{a})}\right)\\
&= \left(4\sqrt{2} + \frac{1}{\sqrt{6}\pi}\right)D^{-\frac{1}{4}}\sum_{[\mathfrak{a}]\in Cl(\mathcal{O}_D)}\sum_{n=1}^{\infty}\exp\left(-\frac{\pi n\sqrt{D}}{N(\mathfrak{a})}\right)\\
&\leq C_1 D^{-\frac{1}{4}}h(-D),
\end{aligned}$$

where

$$C_1 := \left(4\sqrt{2} + \frac{1}{\sqrt{6}\pi}\right)\left(\frac{e^{-\pi\sqrt{3}}}{1 - e^{-\pi\sqrt{3}}}\right).$$

Next, we have

$$\begin{aligned} |S_2| &= \left| \sum_{\alpha\sqrt{D}<N(\mathfrak{a})\leq\sqrt{\frac{D}{3}}} \frac{1}{\sqrt{N(\mathfrak{a})}} \log\left(\alpha\frac{\sqrt{D}}{N(\mathfrak{a})}\right) \right| \\ &\leq \sum_{\alpha\sqrt{D}<N(\mathfrak{a})\leq\sqrt{\frac{D}{3}}} \left| \frac{1}{\sqrt{\alpha}} D^{-\frac{1}{4}} \log\left(\alpha\sqrt{3}\right) \right| \\ &\leq \left| \frac{\log(\alpha\sqrt{3})}{\sqrt{\alpha}} \right| h_{\Omega_D} D^{-\frac{1}{4}} \\ &\leq C_2 D^{-\frac{1}{4}} h_{\Omega_D}, \end{aligned}$$

where

$$C_2 := \frac{\log(\alpha\sqrt{3})}{\sqrt{\alpha}}$$

and

$$h_{\Omega_D} := \#\{z_{\mathfrak{a}} \mid \frac{\sqrt{3}}{2} \leq \mathfrak{Im}(z_{\mathfrak{a}}) \leq \frac{1}{2\alpha}\}.$$

Clearly, $h_{\Omega_D} \leq h(-D)$. On the other hand, since each term in the summand of S_1 is positive, we have (discarding every term except the one with $N(\mathfrak{a}) = 1$)

$$S_1 \geq \log(\alpha\sqrt{D}) \geq \varepsilon \log(D).$$

The second inequality is satisfied for all $D \geq (8\pi/e^{\gamma})^{(\frac{1}{2}-\varepsilon)^{-1}}$. Putting things together, we conclude after a short calculation that

$$h(-D) \geq 0.1265 \cdot \varepsilon D^{\frac{1}{4}} \log(D).$$

□

Remark. Note that by the equidistribution of Heegner points [D] we have

$$\frac{h_{\Omega_D}}{h(-D)} \longrightarrow 1 - \frac{2}{3}\pi\alpha \approx .852$$

as $D \to \infty$.

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