Bounding Components in Real Zero Sets of Bivariate Pentanomials

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1 Introduction

The quest to categorize the possible topological types of algebraic curves has long been of interest to mathematicians. In 1876, Harnack derived $O(n^2)$ bounds on the number of connected components of real algebraic curves of degree n. In 1900, Hilbert proposed his epinomous 16th problem, calling for a categorization of the possible configurations of branches and cyclic components of algebraic curves. Over a century later, this problem remains open for polynomials of degree $n \geq 8$.

Our work takes a slightly different approach, appealing to the study of **sparse** polynomials, or polynomials with relativly few terms. Instead of studying algebraic curves of fixed degree, we are interested in the real zero sets of polynomials with a fixed small number of monomial terms. This allows us to investigate Hilbert-esqe questions for curves of arbitrarily high degree.

Much progress has already been made is this field. A combinatorial method due to Viro (explained in 1.2), gives a categorization of the real zero sets of polynomials in n variables with $(n+1)$ or $(n+2)$ monomial terms, which can be used to derive bounds on the number of connected components.

Our investigation pretains to polynomials in n variables with $(n+3)$ monomial terms, for which the most natural place to start is 2-variate, 5-nomials (**bivariate pentanomials**).

The primary tool we will use in this investigation is an object known as the $\mathcal{A}$-discriminant variety (described in section 2), denoted $\nabla_{\mathcal{A}}$. Informally, given a family $\mathcal{F}$ of polynomials defined by a fixed set of exponent vectors, $\nabla_{\mathcal{A}}$ is the subfamily $\mathcal{F}$ comprising polynomials having a degenerate root (realized in complex coefficient space).

The compliment of this object restricted to real coefficient space often consists of many connected components. Polynomials lying in the same connected component of the compliment (or **chamber** of $\nabla_{\mathcal{A}}$ in real coefficient space will have isotopic real zero sets.

We will see that in the n-variate, $(n+3)$-nomial case, there are certain chambers of the $\mathcal{A}$-discriminant for which Viro's method can be applied. Thus by bounding the change in number of connected components of the real zero sets of polynomials as we cross $\nabla_{\mathcal{A}}$, we can gain information about real zero sets for polynomials in the remaining chambers.

The primary contribution of this report is proving that given $\nabla_{\mathcal{A}}$ for a family $\mathcal{F}$ of bivaiate pentanomials of a certain form, that elements of $\mathcal{F}$ in adjacent chambers of $\nabla_{\mathcal{A}}$ differ in number of connected components by at most one. As a corollary, we give explicit bounds on the number of compact and non-compact connected components of such polynomials.

We state this theorem and corrolary formally after a notational definition.

Definition 1.1. Given a polynomial f, the number of total, compact, and non-compact connected components in the real zero set of f will be denoted $Tot(f)$, $Comp(f)$, and $Non(f)$ respectivly.

Theorem 1.2. *Let $\mathcal{F}$ be an bivariate pentanomial family of the form*

$$\mathcal{F} = \{c_1 + c_2x + c_3y + c_4x^{\alpha_1}y^{\alpha_2} + c_5x^{\beta_1}y^{\beta_2},\ \textit{for fixed } \alpha_i, \beta_i \in \mathbb{Z}\}$$

Given $f, g \in \mathcal{F}$ lying in adjacent chambers of the signed reduced $\mathcal{A}$-discriminant of $\mathcal{F}$, $Non(f) = Non(g)$ and $|Comp(f) - Comp(g)| \leq 1$.

Remark 1.3. Any bivariate pentanomial family can be reduced to a family of the form

$$\mathcal{F} = \{c_1 + c_2x + c_3y + c_4x^{\alpha_1}y^{\alpha_2} + c_5x^{\beta_1}y^{\beta_2},\ \text{for fixed } \alpha_i, \beta_i \in \mathbb{Q},\}$$

we simply restrict to the case where α_1 and β_i are integers (theorem 1.2) or nonnegative integers (corollary 1.4).

Corollary 1.4. *If $f \in \mathcal{F}$, where $\mathcal{F}$ as defined above has $\alpha_i, \beta_i \geq 0$, $Comp(f) \leq 3$ and $Tot(f) \leq 7$.*

This report proceeds as follows. The remainder of this section lays out preliminary definitions and gives an overview of Viro's method and its failing in the $(n+3)$-nomial case. Section 2 gives a detailed exposition of the $\mathcal{A}$-discriminant variety. Section 3 gives a proof of the main theorem (1.2). Section 4 proves the corralary (1.4) and remarks on possibilities for generalization to arbitrary bivariate pentanomials.

1.1 Preliminary Background

We begin with some definitions that will enable us to make connections between the solution sets of n-variate polynomials and certain polygons in $\mathcal{R}^n$.

Definition 1.5. An n-variate $(n+k)$-nomial is a polynomial of the form $f(x_1, \ldots, x_n) = \sum_{i=1}^{n+k} c_i x^{a_i}$ for $c_i \neq 0$. We call $A = \{a_1, ..., a_{n+k}\} \subset \mathbb{Z}^n$ the **support** of f ($Supp(f)$).

Definition 1.6. We say that f is an **honest** n-variate polynomial if its support does not lie in some affine $n-1$ hyperplane.

For example, the polynomial $f = 1 + xy + x^3y^3$ is not an honest bivariate polynomial since it can be realized in only one variable. For the remainder of this paper, we will assume that all of our n-variate polynomials are honest.

Definition 1.7. Given a finite point set $P \subset \mathbb{R}^n$, we define the **convex hull**, $Conv(P)$, to be the minimum convex set $X \subset \mathbb{R}^n$ such that $P \subset X$.

Definition 1.8. The **Newton polygon** of f is defined to be

$$Newt(f) = Conv(\{a_i : a_i \in Supp(f)\}).$$

Assuming that all coefficients of f are real, the **signed Newton polygon** of f ($SNewt(f)$) is given by assigning a sign to each vertex of $Newt(f)$ such that the vertex a_i is given $sign(c_i)$.

1.2 Viro's Method

The topological types of the real zero sets for sufficiently simple polynomial equations can be completely categorized using a combinatorial approach called **Viro's method**. The method gives a combinatorial object which is isomorphic to the positive zero set of a polynomial. For theory underlying this method, refer to [3].

Remark 1.9. While Viro's method in its simplest form categorizes only the positive zero set of a polynomial, it can be extended to apply to the entire real zero set (see section 4).

We first describe the method in the $(n+1)$-nomial case, before discussing its generalizations to the $(n+2)$ and $(n+3)$-nomial cases. Its failing to apply to certain polynomials in this final case is our motivation for this investigation.

Definition 1.10. The Viro diagram of an honest n-variate, $(n+1)$-nomial, $Viro(f)$ is given by $Conv(M)$, where M is the set of midpoints of edges of $SNewt(f)$ whose endpoints have opposite signs.

Example 1.11. *Given $f = 30 - 11x + 7y$, figure 1, shows $Viro(f)$, along with the positive zero set of the polynomial. We notice that, viewing the hypotenuse of $Viro(f)$ as lying at infinity, each object is connected and unbounded at one end.*

In general, the Viro diagram of an honest n-variate $(n+1)$-nomial is diffeomorphic to the classical n-simplex, which exaplains why we need not consider choice of triangulation as we will in higher cases.

Theorem 1.12. *If f is an honest n-variate, $(n+1)$-nomial, $Viro(f)$ is isotopic to the positive zero set of f.*

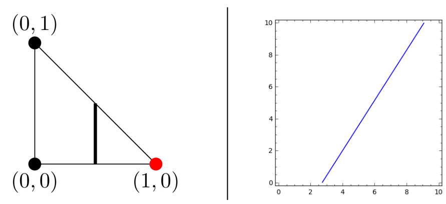


Figure 1: Viro diagram and positive zero set for polynomial in example 1.11.

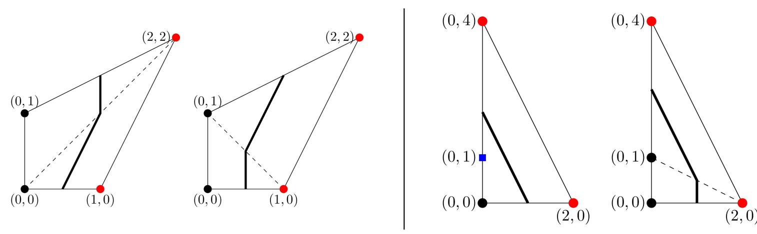


Figure 2: Viro diagrams correcponding to each of the triangulations for $SNewt(g)$ (left) and $SNewt(h)$ (right) from example 1.14

Viro's method in the $(n+2)$-nomial case works similarly, with the caveat that we must first assign a triangulation to $SNewt(f)$.

Definition 1.13. If f is an honest n-variate (n+2)-nomial, then given any triangulation Σ, we define

$$Viro_\Sigma(f) = \bigcup_{\sigma \in \Sigma} Conv(M_\sigma),$$

where M_σ is the set of midpoints of those edges of σ whose endpoints have opposite signs.

Example 1.14. *The signed Newton polygons for $g = 30 - x + 7y - x^2y^2$ and $h = 30 - x + 7y - y^4$ each have two possible trianglations. The Viro diagrams corresponding to each of these is shown in figure 2.*

Theorem 1.15. *If f is an honest n-variate, $(n+2)$-nomial, There exists a trianglation Σ of $SNewt(f)$ such that $Viro_\Sigma(f)$ is isotopic to the positive zero set of f.*

In fact, given any such polynomial, we can use the $\mathcal{A}$-discriminant to determine exactly which triangulation to use. We will not, however, devote any more attention to determining choice of triangulation since it is not relevant to the project at hand.

Unfortunately, theorem 4.2 does not apply in general to the $(n+3)$-nomial case. However, it does extend to the $(n+3)$ case if we require that f correspond to a point in one of a certain subset of the chambers of the $\mathcal{A}$-discriminant variety corresponding to f. An exposition of this enormously useful object will be the focus of the following section.

2 The $\mathcal{A}$-Discriminant Variety

The $\mathcal{A}$-discriminent is a generalization of the more familiar quadratic discriminant. We recall from high school algebra that a polynomial in the univariate trinomial family

$$\mathcal{F} = \{ax^2 + bx + c : a, b, c \in \mathbb{C}^*\}$$

has a degenerate root exactly at the roots of the polynomial

$$\Delta_{(0,1,2)}(a, b, c) = b^2 - 4ac.$$

We will use this familiar example to demonstate the resultants method for computing the $\mathcal{A}$-discriminant polynomial for an n-variate, $(n+k)$-nomial.

We begin with the definition of the $\mathcal{A}$-discriminant variety $\nabla_{\mathcal{A}}$. Simply stated, given family $\mathcal{F}$ with support $\mathcal{A}$, $\nabla_{\mathcal{A}}$ represents the set of polynomials in $\mathcal{F}$ having a nonzero degenerate root.

Definition 2.1. The **$\mathcal{A}$-discriminant variety** of an n-variate $(n+k)$-nomial with support $\mathcal{A} = \{a_1, ..., a_{n+k}\} \subset \mathbb{Z}^n$ is defined as the Zariski closure of:

$$\nabla_{\mathcal{A}} = \{(c_1, ...c_{n+k}) \in P(\mathbb{C})^{n+k-1} :$$
$$\exists \zeta \in (\mathbb{C}^*)^n \text{ with } f(\zeta) = 0 \text{ and } \frac{\partial f}{\partial x_i}(\zeta) = 0 \text{ for all } i \in \{1, ..., n\}\}.$$

Theorem 2.2. *If $f, g \in \mathcal{F}$ correspond to points in the same connected component of the compliment of $\nabla_{\mathcal{A}}$ in real coefficient space, then the real zero sets of f amd g are isotopic.*

2.1 The Resultants Method

Perhaps the most direct way to compute $\nabla_{\mathcal{A}}$ would be to find a polynomial $\Delta_{\mathcal{A}}$ whose solution set is exactly $\nabla_{\mathcal{A}}$. To do this, we will introduce the resultants method for determining whether or not two polynomials have a common root.

Definition 2.3. Given polynomials $f(x) = a_0 + a_1 x + \cdots a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots b_m x^m$, with $a_i, b_i \in \mathbb{C}$ and $a_n, b_m \neq 0$, the **Sylvester matrix** of f and g is defined to be the following $(n+m) \times (n+m)$ matrix:

$$Syl_x(f, g) = \begin{pmatrix} a_0 & a_1 & \cdots & a_m & 0 & 0 \\ 0 & \ddots & & & \ddots & 0 \\ 0 & 0 & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & 0 \\ 0 & \ddots & & & \ddots & 0 \\ 0 & 0 & b_0 & b_1 & \cdots & b_n \end{pmatrix}.$$

A classical theorem describes how we can use the Sylvester matrix to compute $\Delta_{\mathcal{A}}$.

Theorem 2.4. *If $f(x) = a_0 + a_1 x + \cdots a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots b_m x^m$, with $a_i, b_i \in \mathbb{C}$ and $a_n, b_m \neq 0$, then $det(Syl_x(f,g)) = 0$ if and only if $f(\zeta) = g(\zeta) = 0$ for some $\zeta \in \mathbb{C}^*$.*

Thus if $\mathcal{F}$ is a univariate polynomial of degree d with support $\mathcal{A}$, $\Delta_{\mathcal{A}}$ is given by the determinant of the $(2d-1) \times (2d-1)$ matrix $Syl_x(f, f')$.

Example 2.5. *In the case of our quadratic example, we derive the familiar formula, realizing that we may scale $\Delta_{\mathcal{A}}$ by the nonzero constant $-a$.*

$$\Delta_{\mathcal{A}} = det \begin{pmatrix} c & b & a \\ b & 2a & 0 \\ 0 & b & 2a \end{pmatrix} = -a(b^2 - 4ac).$$

This method can also be extended into a method for computing the $\mathcal{A}$-discriminant polynomial for multivariate families.

While the resultants method is practical for this quadratic example, the size and complexity of computing a single Sylvester discriminant grows in the square of the degree of $\mathcal{F}$. Furthermore, $\Delta_{\mathcal{A}}$ for even a sparse polynomial often has an unmanagable number of terms.

We would like an alternate representation for $\nabla_{\mathcal{A}}$ that is both easy to compute and practical to use. We will find such a representation in the Horn-Kapranov uniformization.

2.2 Horn-Kapranov Uniformization

For a family $\mathcal{F}$ with support $A = \{a_1, ..., a_{n+k}\}$, we define an $(n+1) \times (n+k)$ matrix

$$\hat{A} = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n+k} \end{pmatrix}$$

and a corresponding $(n+k) \times (k-1)$ matrix B whose columns form a basis of the right null of $\hat{A}$:

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_{n+k} \end{pmatrix}.$$

Theorem 2.6. *∇_A can be parametrized in $P(\mathbb{C})^{n+k-1}$ as the Zariski closure of the following:*

$$\varphi(\nabla_A) = \{(b_1 \cdot \lambda) t^{a_1} : \cdots : (b_{n+k} \cdot \lambda) t^{a_{n+k}} \mid \lambda \in P(\mathbb{C})^{k-2}\}.$$

We will denote the ith component of this parametriztion as γ_i.

Example 2.7. *Consider the family*

$$\mathcal{F} = c_1 + c_2 x + c_3 y + c_4 x^4 y + c_5 x y^4.$$

Using

$$\hat{A}B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -4 & 3 \\ -1 & -3 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = 0,$$

we have

$$\varphi(\nabla_A) = \{(4\lambda_1 t^{(0,0)} : (-4\lambda_1 + 3\lambda_2) t^{(1,0)} : (-\lambda_1 - 3\lambda_2) t^{(0,1)} : (\lambda_1 - \lambda_2) t^{(4,1)} : \lambda_2 t^{(1,4)}) : \lambda \in P(\mathbb{C})\}$$

While $\varphi(\nabla_A)$ is easy to compute, its dimension depends on the number of monomial terms in $\mathcal{F}$. In order to visualize $\varphi(\nabla_A)$ in a lower dimension, we will take advantage of homogenaties to reduce $\mathcal{F}$ to a subfamily with coefficient space of dimension $k-1$. We will also restrict our attention to the real component of the coefficient space.

We observe that if $\zeta \in (\mathbb{C}^*)^n$, $f(x_1, \ldots, x_n)|_\zeta = 0$ if and only if $\kappa f(\alpha_1 x_1, \ldots, \alpha_n x_n)|_\zeta = 0$.. Thus we may study the behavior of a subfamily of polynomials that will represent equivalence classes of $\mathcal{F}$.

Keeping $\mathcal{F}$ as in example 2.7, consider the subfamily $\mathcal{F}' = 1 + x + y + ax^4y + bxy^4$, which is the image of the map

$$[\cdot] : \mathcal{F} \to \mathcal{F}' \text{ by } [f(x,y)] = \frac{1}{c_1} f\left(\frac{c_1}{c_2} x, \frac{c_1}{c_3} y\right),$$

which in turn induces a map on the coefficient vectors

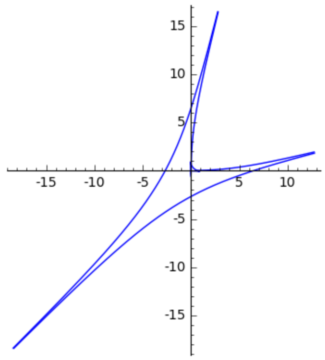
$$[\cdot]_* : P(\mathbb{C})^5 \to (\mathbb{C}^*)^2 \text{ by } [(c_1 : c_2 : c_3 : c_4 : c_5)]_* = \left(\frac{c_1^4 c_4}{c_2^4 c_3}, \frac{c_1^4 c_5}{c_2 c_3^4}\right).$$

In order to utilize the reduced $\mathcal{A}$-discriminant, we will find it useful to consider the ***amoeba*** *of its* ***contour****, meaning that we will plot the log-norm of its real component.*

The reduced $\mathcal{A}$-discriminant for our example with respect to the change of variables $[\cdot]$ is the following:

$$\overline{\varphi}(\nabla_A) = \left(log\left|\frac{\gamma_1^4\gamma_4}{\gamma_2^4\gamma_3}\right|, log\left|\frac{\gamma_1^4\gamma_5}{\gamma_2\gamma_3^4}\right|\right) =$$

$$\left\{\left(log\left|\frac{(4\lambda_1)^4(\lambda_1-\lambda_2)}{(-4\lambda_1+3\lambda_2)^4(-\lambda_1-3\lambda_2)}\right|,\ log\left|\frac{(4\lambda_1)^4(\lambda_2)}{(-4\lambda_1+3\lambda_2)(-\lambda_1-3\lambda_2)^4}\right|\right) : \lambda \in P(\mathbb{C})\right\}$$



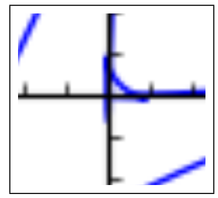


Figure 3: Reduced $\mathcal{A}$-discriminant amoeba for $\mathcal{F}$ (right image zoomed in to show cusps). In addition to the 3 clearly visible places where the discriminant 'blows up', the discriminant also extends infinitely along the negative x and negative y axes.

2.3 Unfolding the Reduced $\mathcal{A}$-Discriminant

We would like to be able to apply theorem 2.2 to $\overline{\varphi}(\nabla_A)$, however in taking the log norm, $\overline{\varphi}(\nabla_A)$ is no longer homeomorphic to ∇_A. To preserve the bijection, we instead take the log norm of each quadrant of the coefficient space separately.

Definition 2.8. Let $\mathcal{F}$ be a family of n-variate, $(n+k)$-nomials.

For $(\sigma_1, \dots, \sigma_{k-1}) \in \{\pm 1\}^{k-1}$, then we define

$$\overline{\varphi}(\nabla_A)_{(\sigma_1,\dots,\sigma_{k-1})} = \overline{\varphi}(\nabla_A)|_{\lambda_{(\sigma_1,\dots\sigma_{k-1})}},$$

where

$$\lambda_{(\sigma_1,\dots\sigma_{k-1})} = \{\lambda \in P(\mathbb{R})^{k-1} : \frac{\hat{\varphi}_i(\lambda)}{|\hat{\varphi}_i(\lambda)|} = \sigma_i \text{ for } i \in \{1,\dots,k-1\}.,$$

and $\hat{\varphi}$ is the map identical to $\overline{\varphi}$ except that we refrain from taking the log norm.

We observe that $\overline{\varphi}$ is undefined exactly where $b_i \cdot \lambda = 0$ for some row b_i of B. Thus in the $(n+3)$-nomial case, it is undefined for at most $(n+3)$ distinct values of $\lambda \in P((R))$- near these values, the function goes off to infinity in some direction. Thus $\overline{\varphi}$ is the union of ≤ 5 (smooth) unbounded connected components. Since in between these values, $\overline{\varphi}$ is continuous (and nonzero), each of its connected components is contained completely in $\overline{\varphi}(\nabla_A)_{(\sigma_1,\dots,\sigma_{k-1})}$ for some $\sigma_1, \dots, \sigma_{k-1}$.

It is often helpful to visualize $P((R))$ as the upper half circle (with the 2 endpoints identified).

$$\begin{pmatrix} 4 & 0 \\ -4 & 3 \\ -1 & -3 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

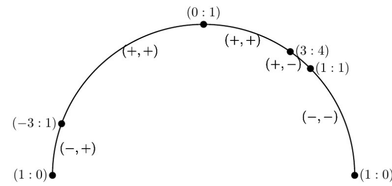


Figure 4: B matrix and annotated domain for example 2.9

Example 2.9. *Continuing again our example*

$$\mathcal{F} = 1 + x + y + ax^4y + bxy^4,$$

figure 4 shows the domain of $\overline{\varphi}$. The marked points represent the values of λ for which $\overline{\varphi}$ 'blows up', and the ordered pairs of signs denote which quadrant of the reduced $\overline{\varphi}$ contains the image of this portion of the domain.

Figure 5 shows the signed reduced $\mathcal{A}$-discriminant for mathcal$F$.

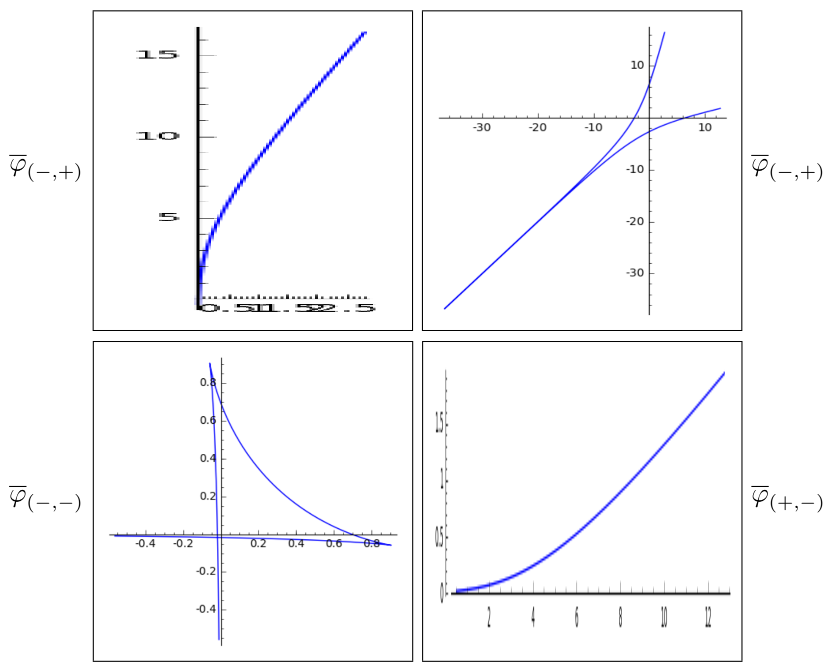


Figure 5: The signed reduced $\mathcal{A}$-discriminant corresponding to example 2.9.

Having established in detail the machinary needed to work with the $\mathcal{A}$-discriminant, we proceed to prove the main theorem regarding change in number of comnected compoents of a polynomial as we cross between chambers.

3 Proof of Main Theorem

The proof of the theorem is intuitive and the following section mainly amounts to an untangling of subtleties - the outline of the argument is as follows.

Consider a family $\mathcal{F}$ of the form described in theorem 1.2 (main theorem).

1. *Given a polynomial f on the boundary between two chambers of the $\mathcal{A}$-discriminant for $\mathcal{F}$, there exists constant ϵ such that $f+\epsilon$ and $f-\epsilon$ lie in opposite chambers.*

2. *We can ensure that f has exactly one degenerate root and that at this root, the surface defined by $f(x,y)$ attains either a local extremum or a saddle point.*

3. *Assuming that ϵ is sufficiently small, the cross sections of the surface $f(x,y)$ at $f(x,y) = \pm\epsilon$ differ in number of (compact) connected components by at most 1.*

Definition 3.1. We define the set of critical points of surface $f(x,y)$ to be $W = \{(x,y) \in \mathbb{R}^2 : \frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0\}$. Define the z-coordinate projection $\pi_z : W \to \mathbb{R}$ to be $\pi_z(x,y) = f(x,y)$. We refer to the image $\pi_z(W)$ as the set of **critical values** of f.

Lemma 3.2. *A surface $f(x,y)$, where $f(x,y) = 0$ is a bivariate polynomail, has finitly many critical values, ie $\pi_z(W)$ is finite.*

Proof. By Sard's theorem, we have that $\pi_x(W)$ has Lebesgue measure zero in $\mathbb{R}$. We now appeal to the theory of semi-algebraic sets (a superset of algebraic sets). The projection of a semialgebraic set is semialgebraic, and a semialgebraic set is the finite union of points and open intervals. Thus a measure zero semi-algebraic set is a finite point set.

□

For a surface $f(x,y)$, let $\delta_{f(x,y)} = min\{|w| : w \in \pi_z(W)\}$.

Remark 3.3. Since topological type is constant within a chamber of the signed $\mathcal{A}$-discriminant, it suffices to show that given any pair of adjacent chambers, the result holds for some pair of polynomials on opposite sides.

3.1 Generic properties of polynomials on $\overline{\varphi}(\nabla_{\mathcal{A}})$

Consider the surface defined by $f(x,y) = 1 + x + y + ax^{\alpha_1}y^{\alpha_2} + bx^{\beta_1}y^{\beta_2}$. According to the definition of $\pi_z(W)$, (x,y) is a degenerate root of f exactly when $(x,y,f(x,y) = 0)$ is a critical point. Henceforth we will refer to degenerate roots as ***critical roots*** *to premptivly avoid confusion with degenerate critical points.*

The critical points of an algebraic surface in $\mathbb{R}^3$ come in several forms. A nondegenerate critical point is either a local extremum (maximum or minimum) *or a saddle point. Critical points can be categorized by studying the eigenvalues of the* ***Hessian matrix****.*

Definition 3.4. Given a c_2 function $f : \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix of f is defined as follows:

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Definition 3.5. A critical point $x \in \mathbb{R}^n$ is degenerate if

$$det(H(f)|_x) = 0,$$

ie the Hessian is neither positive semidefinite nor negative semidefinite.

Before discussing generic properties, we define this notion of a generic property.

Definition 3.6. We say a property P is **generic** in $\mathbb{R}^n$ if it holds on the compliment of an algebraic set - ie there exists an honest n-variate polynomial f such that P holds exactly when $f = 0$.

By showing that polynomials on the $\mathcal{A}$-discriminant generically have nondegenerate critical roots, we will be able to assume without loss of generality that a critical root of such a polynomial is either a local extremum or a saddle point.

Theorem 3.7. *Given family*

$$\mathcal{F} = \{1 + x + y + ax^{\alpha_1}y^{\alpha_2} + bx^{\beta_1}y^{\beta_2}\},$$

where α_i and β_i are fixed integers, a generic point on $\overline{\varphi}(\nabla_{\mathcal{A}})$ has exactly one critical root. Furthermore, this critical root is a nondegenerate critical point.

Proof. $\mathcal{F}$ has $\hat{A}$ matrix

$$\hat{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & 1 & \alpha_2 & \beta_2 \end{pmatrix},$$

whose null space is given by

$$\begin{pmatrix} 1-\alpha & 1-\beta \\ \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$,

and Hessian matrix

$$H(f) = \begin{pmatrix} a\alpha_1^2 x^{\alpha_1 - 2} y^{\alpha_2} + b\beta_1^2 x^{\beta_1 - 2} y^{\beta_2} & a\alpha_1\alpha_2 x^{\alpha_1 - 1} y^{\alpha_2 - 1} + b\beta_1\beta_2 x^{\beta_1 - 1} y^{\beta_2 - 1} \\ a\alpha_1\alpha_2 x^{\alpha_1 - 1} y^{\alpha_2 - 1} + b\beta_1\beta_2 x^{\beta_1 - 1} y^{\beta_2 - 1} & a\alpha_2^2 x^{\alpha_1} y^{\alpha_2 - 2} + b\beta_2^2 x^{\beta_1} y^{\beta_2 - 2} \end{pmatrix}.$$

Recalling our definition of the $\mathcal{A}$-discriminant, a point on $\overline{\varphi}(\nabla_{\mathcal{A}})$ which is the image of $\lambda \in P(\mathbb{R})$ satisfies the following equation:

$$c * \begin{pmatrix} 1 \\ x \\ y \\ ax^{\alpha_1} y^{\alpha_2} \\ bx^{\beta_1} y^{\beta_2} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 - \alpha \\ \alpha_1 \\ \alpha_2 \\ -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 - \beta \\ \beta_1 \\ \beta_2 \\ 0 \\ -1 \end{pmatrix},$$

where c is a real scaler accounting for the fact that λ lies in projective space, (x, y) is a critical root, and a, b are real coefficients.

Since $P(\mathbb{R})$ can be represented as the set of lines through the origin in $\mathbb{R}^2$, we can identify λ by the ratio $\frac{\lambda_2}{\lambda_1}$. We begin by fixing $\lambda_2 = r\lambda_1$, eg $\lambda_1 = 1$, $\lambda_2 = r$, and following though a series of implications.

$\rightarrow$ $c = 1 - \alpha + r(1 - \beta)$ depends linearly on r.

$\rightarrow$ $x = c^{-1}(\alpha_1 + r\beta_1)$, and similarly y, is determined by r.

$\rightarrow$ $a = c^{-1}(\frac{-1}{x_1^{\alpha} y_2^{\alpha}})$, and similarly b, is also determined by r.

Both portions of the theorem follow immediately from these relations.

Since (x, y) is determined by λ, a polynomial has multiple critical roots only if it corresponds to a self intersection of $\overline{\varphi}$. Since all intersections of $\overline{\varphi}$ are transverse (see [5]), a generic point on $\overline{\varphi}$ has only one critical root.

Furthermore, assuming that $\lambda \cdot b_i = 0$ for all rows b_i (otherwise $\overline{\varphi}$ is not defined), $det(H(f)|_{x,y})$ is a univariate polynomial equation in variable r. Thus, generically a critical root is nondegenerate.

□

3.2 The Proof

Let $\mathcal{F} = \{c_1 + c_2 x + c_3 y + c_4 x^{\alpha_1} y^{\alpha_2} + c_5 x^{\beta_1} y^{\beta_2} : c_i \in \mathcal{R}\}$ *and suppose* $\overline{\varphi}(\nabla_{\mathcal{A}}) \in \mathbb{R}^2$ *is the reduction of* $\nabla_{\mathcal{A}}$ *to the subfamily* $\mathcal{F}' = \{1 + x + y + ax^{\alpha_1} y^{\alpha_2} + bx^{\beta_1} y^{\beta_2} : a, b \in \mathbb{R}\}$. *by the followinf map:*

$$[\cdot] : \mathcal{F} \rightarrow \mathcal{F}' \text{ by } [f(x, y)] = \frac{1}{c_1} f(\frac{c_1}{c_2} x, \frac{c_1}{c_3} y).$$

Definition 3.8. If quadrant $\overline{\varphi}(\nabla_{\mathcal{A}})_{\sigma_1,\sigma_2}$ of the unfolded reduced $\mathcal{A}$-discriminant has chambers $\{V_1, \cdots V_n\}$, then chambers V_i and V_j are **adjacent** if $\overline{V}_i \cap \overline{V}_j$ contains the injective image of an interval. We will call $\overline{V}_i \cap \overline{V}_j$ the **common boundary** of V_i and V_j.

In order to show that the theorem hold for all $f, g : f \in V_i, g \in V_j$, we begin by considering a polynomial

$$f = 1 + x + y + \sigma_1 e^{a_0} x^{\alpha_1} y^{\alpha_2} + \sigma_2 e^{b_0} x^{\beta_1} y^{\beta_2} \tag{2}$$

corresponding to a point p=(a_0, b_0) on $\overline{\varphi}(\nabla_{\mathcal{A}})_{\sigma_1,\sigma_2}$. We can assume without loss of generality that f has exactly one critical root and that $\frac{\partial \overline{\varphi}_2}{\partial \overline{\varphi}_1}|_p \neq \frac{\beta-1}{\alpha-1}$ (the second fact is clearly true since the second derivative of $\overline{\varphi}$ are generically nonzero).

We consider the sum of f with the constant monomial ϵ, where we assume $|\epsilon| < min(1, \delta_f)$. Using our change of variables [$\cdot$],

$$[f + \epsilon] = 1 + x + y + \sigma_1 e^{a_0}(1+\epsilon)^{\alpha-1} x^{\alpha_1} y^{\alpha_2} + \sigma_2 e^{b_0}(1+\epsilon)^{\beta-1} x^{\beta_1} y^{\beta_2}. \tag{3}$$

Then the parametric curve of points corresponding to $[f + \epsilon]$ is

$$C = (log(e^{a_0}(1+\epsilon)^{\alpha-1}), log(e^{b_0}(1+\epsilon)^{\beta-1})), \tag{4}$$

with constant slope $\frac{\partial C_2}{\partial C_1} = \frac{\beta-1}{\alpha-1}$.

Thus C crosses $\overline{\varphi}$ transversely at p, and for ϵ of sufficiently small absolute value, we obtain $[f + \epsilon] \in V_i, [f - \epsilon] \in V_j$.

It remains only to examine the change in number of connected components as the cross section for $f(x, y)$ changes from $f + \epsilon = 0$ to $f - \epsilon = 0$, assuming f has a single nondgenerate critical root and the surface $f(x, y)|_{-\epsilon}^{\epsilon}$ has no other critical points.

It is a result of classical Morse theory that the topological type of the cross section of a surface changes only at critical values (for more on CMT see e.g. [2]. Thus the cross sections of the restriction of $f(x, y)$ to the domain with an open neighborhood around the critical root are all isotopic.

An analysis of directional second derivatives illuminates the behavior around the critical root. The behavior around a nondegenerate critical point is illustrated in figure 3.2.

If the root is a local extremum, varying the cross section in one direction causes the genesis of a compact component, while varying it in the other direction does the reverse. Thus around a local extremum, $Non(f)$ remains constant and $Comp(f)$ changes by exactly 1.

At a saddle point, taking cross-scetions ϵ above and below the critical point locally give 2 distinct components that change their relative configuration as shown in 3.2. $Non(f)$ stays constant and $Comp(f)$ changes by 0 or 1 depending on whether the two components are globally part of the same connected component of the curve.

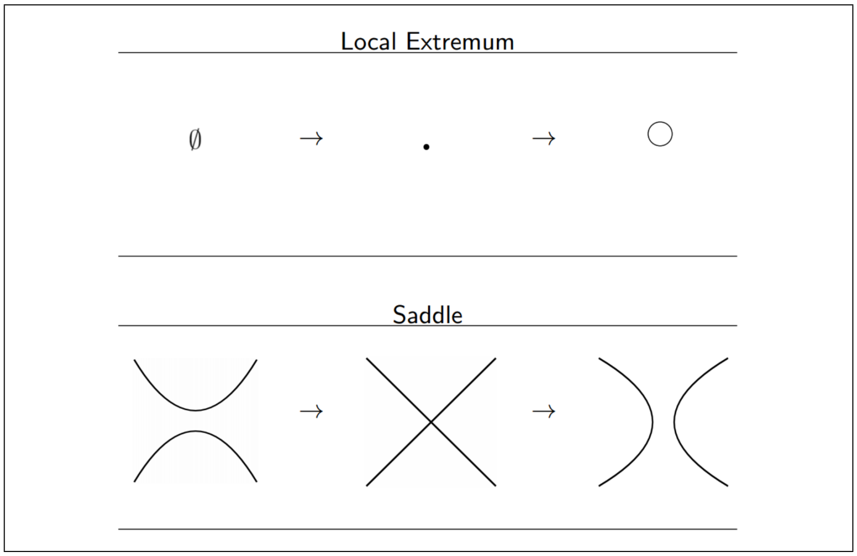


Figure 6: Change in local isotopy type of cross sections of $f(x, y)$ as we cross a critical value.

4 Viro's Method and Bounds on Connected Components

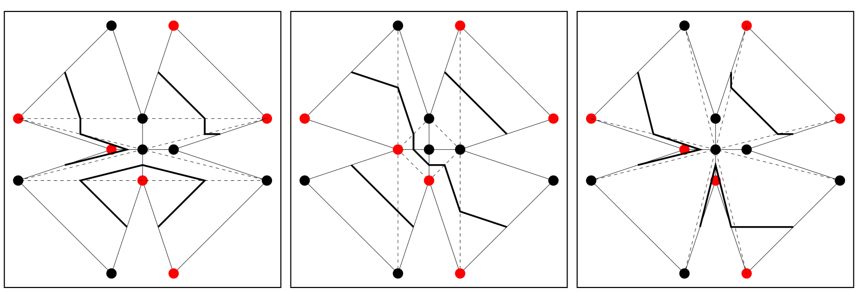


Figure 7: Viro diagrams for $\mathcal{F}_{(-,-)}$ using 3 different triagulations

4.1 Bounds on Components for polynomials in Outer Chambers

We recall from 1.2 that Viro's method applies to $(n+3)$-nomials in certain chambers of the $\mathcal{A}$-discriminant. We now make this more precise.

Definition 4.1. In the setting of $(n+3)$-nomials, an **outer** chamber is a connected components of $\overline{\varphi}$ having unbounded area. A non-outer chamber is an

inner chamber.

Theorem 4.2. *If f is an honest n-variate, $(n+3)$-nomial lying in an outer chamber of the corresponding $\mathcal{A}$-discriminant variety, there exists a trianglation Σ of $SNewt(f)$ such that $Viro_\Sigma(f)$ is isotopic to the positive zero set of f.*

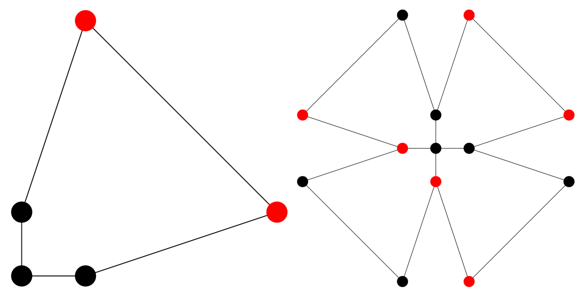


Figure 8: Classical and expanded signed Newton Polygon for $\mathcal{F}$.

So to bound the number of components of real zero sets of certain families of bivariate pentanomials, we first bound the number of components for polynomials in the outer chambers, and the bound the 'depth' of an arbitrary chamber.

We expand slightly on the version of Viro's method described in section 1.2 for the case where the exponents in our polynomial are positive. Instead of simply using the signed Newton polygon to characterize just the positive zero set, we reflect $SNewt(f)$ across both the x and y axes, flipping the signs acordingly based on the powers of x and y. We then assign a triagulation to the $SNewt$, and reflect this triangulation to the additional quadrants, constructing the Viro diagram as before. In this way we characterize the entire real zero set.

Figure 8 shows the classical and extended signed Newton polygons for

$$\mathcal{F}_{(-.-)} = \{1 + x + y - |a|x^4y - |b|xy^4\}.$$

Figure 7 gives the Viro diagrams corresponding to $\mathcal{F}_{(-,-)}$ under three distinct triangulations. By symmetry, these are comprehensive. From this we see that each polynomial in the outer chamber of $\overline{\varphi}_{(-,-)}$ has three, unbounded connected components. We also note that the number of unbounded connected components does not depend on choice of triangulation, but rather on the number of sign alternations on the outer boundary of the extended $SNewt$.

The shape of the (extended) SNewt depends on the support of $\mathcal{F}$, where as before we fix the first three support vectors to be $((0,0),(1,0),(0,1))$, with the remaining exponents being positive integers. Figure 9 illustrates each of the five fundamentally different shapes for the Newton polygon and the maximum number of components for a corresponding Viro diagram. The blue vertices are those that can be designated either positive of negative.

We summarize this figure in the following lemma.

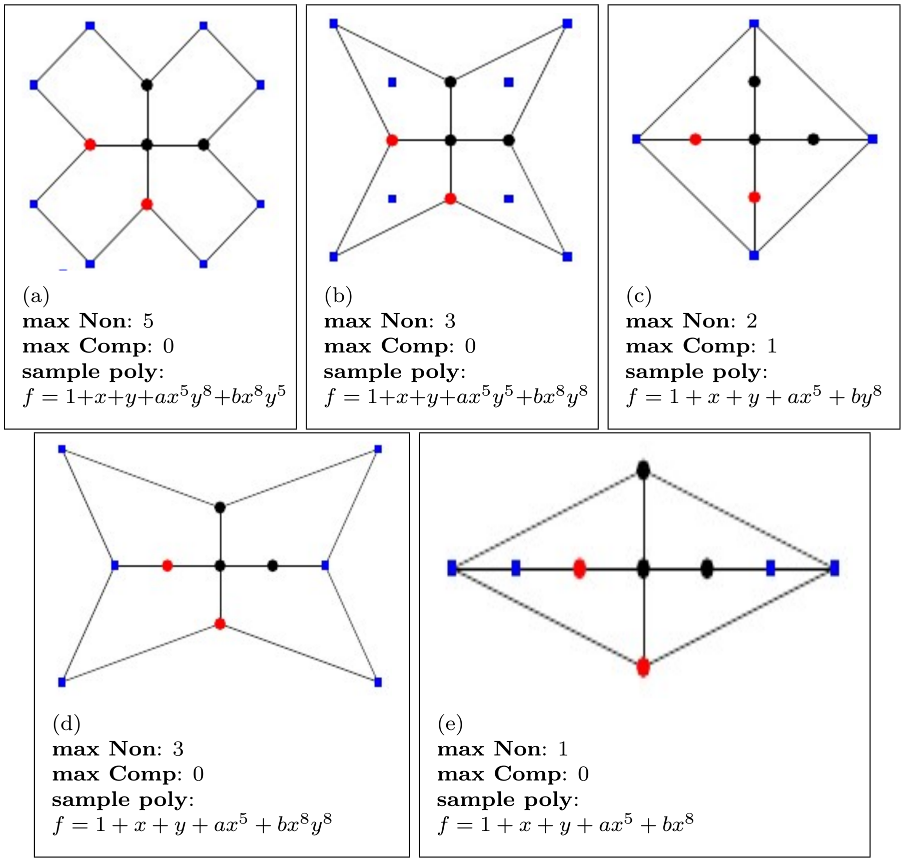


Figure 9: Maximum components by shape of Newton Polygon

Lemma 4.3. *Given a polynomial* $f \in \{1+x+y+ax^{\alpha_1}y^{\alpha_2}+bx^{\beta_1}y^{\beta_2}\}$, *for fixed* $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$,

$$Non(f) \leq 5,\ Comp(f) \leq 1,\ \textit{and } Tot(f) \leq 5.$$

4.2 Proof of Corollary

*By the **depth** of a chamber* V_i, *we mean the minimum length of a path* $V_{outer}, \ldots, V_i$, *where* V_{outer} *ranges over the outer chambers and sucsessive chambers are required to be adjacent. E.g. an outer chamber has depth 1. We now prove corollary 1.4.*

Consider again family $\mathcal{F}$ *as in lemma 4.3. Then we have:*

$$\hat{\varphi} = \left(\frac{\gamma_1^{\alpha-1}\gamma_4}{\gamma_2^{\alpha_1}\gamma_3^{\alpha_2}}, \frac{\gamma_1^{\beta-1}\gamma_5}{\gamma_2^{\beta_1}\gamma_3^{\beta_2}} \right),$$

where γ_i are linear form on $\lambda \in P(\mathbb{R})$.

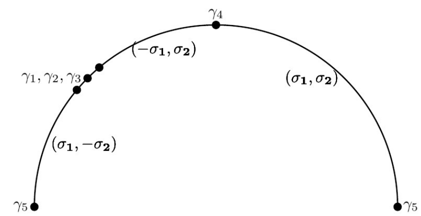


Figure 10: Domain for $\mathcal{F}$ with some arcs marked with quadrant of their image in signed $\overline{\varphi}$. $\overline{\varphi}$ is with respect to B as above.

When $\gamma_4 = 0$, $\overline{\varphi}$ is undefined. Also, since γ_4 has multiplicity 1 in $\hat{\varphi}_1$, this quantity changes sign to either side of the value of λ associated with $\gamma_4 = 0$. Thus the portions of the domain on either side of $\gamma_4 = 0$ lie in distinct quadrants of the signed $\mathcal{A}$-discriminant. The same reasoning applies to γ_5 with regards to $\hat{\varphi}_2$.

If we verify that the zeros for γ_4 and γ_5 are adjacent in the domain, we ensure that at least 3 quadrants of signed $\overline{\varphi}$ are non-empty. This would also imply that if a given quadrant contains the images of exactly 3 arcs, then these arcs are adjacent. Using the same B matrix as in section 3, this is indeed the case. Figure 10 clarifies this notion.

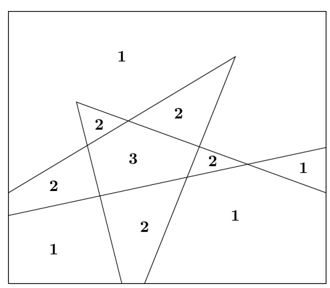


Figure 11: Extremal configuration of arcs in quadrant of $\overline{\varphi}$. Chambers are labelled with depth.

Having established that a given quadrant contains at most three arcs, it is straightforward to charactarize the type of extremal example that maximize chamber depth. To do this, we recall several following results from [5]:

1. *The reduced $\mathcal{A}$ discriminant for an n variate $(n+3)$-nomial has at most 2 cusps. We partition each arc into sub-arcs at the cusps. We then have, in the extremal case, 5 sub-arcs.*

2. *A subarc has no points of self-intersection.*

3. *Adjacent sub-arcs cannot intersect.*

These 3 facts bound the number of intersections in the extremal case to at most 6. Figure 11 illustrates the general form of a chamber containing 3 arcs and 5 sub-arcs. From this it is combinatorially evident that the depth of an arbitrary chamber does not exceed 3. It is easy to check that the non-extremal cases do not affect this bound.

Combined with the lemma from 4.1, this gives us the corollary.

4.3 Some Final Remarks

1. *The author suspects that that the processes used to prove the theorem and corollary, if not the results themselves, could with minimal modification be applied to arbitrary families of bivariate pentanomials. In particular, an arbitrary family of bivariate pentanomials is equivilant to a family of the form*

$$\mathcal{F} = \{1 + x^{\kappa_1} + y^{\kappa_2} + ax^{\alpha_1}y^{\alpha_2} + bx^{\beta_1}y^{\beta_2}\}.$$

   for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$.

2. *We are not certain whether our bound on the number of compact (or total) components is sharp. Finding extremal examples with 2 or 3 compact components would be a worthwhile pursuit. The previous subsection's analysis of sign changes across the domain provides some insight into where to look.*

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