Nonvanishing of Hecke L-Series and ℓ-torsion in Class Groups

Arianna Iannuzzi, Alex Mathers, and Maria Ross

July 17, 2017

Introduction

Alex Mathers

July 17, 2017

Group Characters

- Let G be an abelian group. A *character* of G is a homomorphism $\chi : G \to \mathbb{C}^*$.

Group Characters

- Let G be an abelian group. A *character* of G is a homomorphism $\chi : G \to \mathbb{C}^*$.
- The set of characters of G form a group.

Group Characters

- Let G be an abelian group. A *character* of G is a homomorphism $\chi : G \to \mathbb{C}^*$.
- The set of characters of G form a group.
- A *Dirichlet character of modulus m* is a group character for $G = (\mathbb{Z}/m\mathbb{Z})^*$, or equivalently a multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

$$\text{(i) } \chi(n+m) = \chi(n) \text{ for all } n,$$

$$\text{(ii) } \chi(n) = 0 \text{ for } \gcd(n,m) > 1.$$

L-series

- If χ is a Dirichlet character, then the *L-series* of χ is defined by the series

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

L-series

- If χ is a Dirichlet character, then the *L-series* of χ is defined by the series

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

- Example: The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation

- The L-series of a Dirichlet character χ extends to a meromorphic function on the entire complex plane via *analytic continuation.*

Functional Equation

- The L-series of a Dirichlet character χ extends to a meromorphic function on the entire complex plane via *analytic continuation*.
- This analytic continuation satisfies a *functional equation* of the form $s \mapsto 1 - s$ with central value $L(\chi, 1/2)$.

Functional Equation

- The L-series of a Dirichlet character χ extends to a meromorphic function on the entire complex plane via *analytic continuation.*
- This analytic continuation satisfies a *functional equation* of the form $s \mapsto 1 - s$ with central value $L(\chi, 1/2)$.
- Example: If we let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, then we have the functional equation

$$\xi(s) = \xi(1 - s)$$

and the central value is given by $\zeta(1/2)$.

Our “set up”

- Fix a triple of integers (d, k, D) satisfying:
  - $d \equiv 1 \pmod 4$,
  - $k > 0, \quad \mathrm{sign}(d) = (-1)^{k-1}$,
  - $D > 0, \quad D \equiv 7 \pmod 8, \quad \gcd(d, D) = 1.$

Our "set up"

- Fix a triple of integers (d, k, D) satisfying:
  - $d \equiv 1 \pmod 4$,
  - $k > 0, \quad \text{sign}(d) = (-1)^{k-1}$,
  - $D > 0, \quad D \equiv 7 \pmod 8, \quad \gcd(d, D) = 1$.
- Let K be the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$.

The Class Group

- If $\mathcal{O}_K$ denotes the ring of integers of K, then K can be considered as an $\mathcal{O}_K$-module. Denote the set of *fractional ideals* of $\mathcal{O}_K$ by I_K.

The Class Group

- If $\mathcal{O}_K$ denotes the ring of integers of K, then K can be considered as an $\mathcal{O}_K$-module. Denote the set of *fractional ideals* of $\mathcal{O}_K$ by I_K.
- The set I_K is an abelian group under multiplication; denote the subgroup of "principal" ideals by P_K, and set

$$\mathrm{Cl}(K) = I_K/P_K.$$

The Class Group

- If $\mathcal{O}_K$ denotes the ring of integers of K, then K can be considered as an $\mathcal{O}_K$-module. Denote the set of *fractional ideals* of $\mathcal{O}_K$ by I_K.
- The set I_K is an abelian group under multiplication; denote the subgroup of "principal" ideals by P_K, and set

$$\mathrm{Cl}(K) = I_K/P_K.$$

- This is called the *class group*. It is finite, and its order (the *class number*) is denoted $h(-D)$.

Canonical Hecke Characters

- A *canonical Hecke character* for some "distinguished subgroup" I_D of I_K is, roughly speaking, a character $\psi_k : I_D \to \mathbb{C}^*$ which can be decomposed into a "finite part" and "infinite part", and satisfies

$$\psi_k((\alpha)) = \pm\alpha^{2k-1} \text{ for } (\alpha, \sqrt{-D}\mathcal{O}_K) = 1.$$

Canonical Hecke Characters

- A *canonical Hecke character* for some "distinguished subgroup" I_D of I_K is, roughly speaking, a character $\psi_k : I_D \to \mathbb{C}^*$ which can be decomposed into a "finite part" and "infinite part", and satisfies

$$\psi_k((\alpha)) = \pm\alpha^{2k-1} \text{ for } (\alpha, \sqrt{-D}\mathcal{O}_K) = 1.$$

- Given such a ψ_k, we can define its "quadradic twist" $\psi_{d,k}$. We denote the set of all $\psi_{d,k}$ by $\Psi_{d,k}(D)$; there are exactly $h(-D)$ such characters.

Hecke L-series

- To a canonical Hecke character $\psi \in \Psi_{d,k}(D)$, we can assign an L-series $L(\psi, s)$, which converges for $\mathrm{Re}(s) > k + \frac{1}{2}$.

Hecke L-series

- To a canonical Hecke character $\psi \in \Psi_{d,k}(D)$, we can assign an L-series $L(\psi, s)$, which converges for $\mathrm{Re}(s) > k + \frac{1}{2}$.
- This Hecke L-series has an analytic continuation satisfying a functional equation of the form $s \mapsto 2k - s$,

$$L(\psi, s) = L(\psi, 2k - s),$$

Hecke L-series

- To a canonical Hecke character $\psi \in \Psi_{d,k}(D)$, we can assign an L-series $L(\psi, s)$, which converges for $\mathrm{Re}(s) > k + \frac{1}{2}$.
- This Hecke L-series has an analytic continuation satisfying a functional equation of the form $s \mapsto 2k - s$,

$$L(\psi, s) = L(\psi, 2k - s),$$

- We are interested in the central value $L(\psi, k)$, specifically in determining whether it is zero or nonzero.

Arithmetic Significance

- Let $d = k = 1$. Then our characters $\psi \in \Psi_{1,1}(D)$ naturally correspond to canonical examples of Gross's $\mathbb{Q}$-curves over $K = \mathbb{Q}(\sqrt{-D})$. If $A(D)$ is such an elliptic curve, then

$$L(A(D), s) = \prod_{\psi \in \Psi_{1,1}(D)} L(\psi, s).$$

Arithmetic Significance

- Let $d = k = 1$. Then our characters $\psi \in \Psi_{1,1}(D)$ naturally correspond to canonical examples of Gross's $\mathbb{Q}$-curves over $K = \mathbb{Q}(\sqrt{-D})$. If $A(D)$ is such an elliptic curve, then

$$L(A(D), s) = \prod_{\psi \in \Psi_{1,1}(D)} L(\psi, s).$$

- If $L(\psi, 1) \neq 0$ for some $\psi \in \Psi_{1,1}(D)$, then $L(\psi, 1) \neq 0$ for all $\psi \in \Psi_{1,1}(D)$, hence $L(A(D), 1) \neq 0$.

Arithmetic Significance

- Let $d = k = 1$. Then our characters $\psi \in \Psi_{1,1}(D)$ naturally correspond to canonical examples of Gross's $\mathbb{Q}$-curves over $K = \mathbb{Q}(\sqrt{-D})$. If $A(D)$ is such an elliptic curve, then

$$L(A(D), s) = \prod_{\psi \in \Psi_{1,1}(D)} L(\psi, s).$$

- If $L(\psi, 1) \neq 0$ for some $\psi \in \Psi_{1,1}(D)$, then $L(\psi, 1) \neq 0$ for all $\psi \in \Psi_{1,1}(D)$, hence $L(A(D), 1) \neq 0$.
- By known results towards the BSD conjecture, this implies that the rank of $A(D)$ is zero, and hence the group of K-rational points is finite.

Statement of Results

Arianna Iannuzzi

July 17, 2017

Outlining our Goals

- Since $\#\Psi_{d,k}(D) = h(-D)$, by Siegel's theorem

$$h(-D) \gg_{\epsilon} D^{\frac{1}{2}-\epsilon}$$

we have

$$\#\Psi_{d,k}(D) \gg_{\epsilon} D^{\frac{1}{2}-\epsilon}.$$

Outlining our Goals

- Since $\#\Psi_{d,k}(D) = h(-D)$, by Siegel's theorem

$$h(-D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}$$

we have

$$\#\Psi_{d,k}(D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}.$$

- We would like to quantify the number of $\psi \in \Psi_{d,k}(D)$ with nonvanishing central value. Therefore we define

$$NV_{d,k}(D) = \#\{\psi \in \Psi_{d,k}(D) : L(\psi, k) \neq 0\}.$$

Outlining our Goals

- We would like to find a bound of the form

$$NV_{d,k}(D) \gg D^{\delta_k}$$

for some $\delta_k > 0$.

‘

Outlining our Goals

- We would like to find a bound of the form

$$NV_{d,k}(D) \gg D^{\delta_k}$$

  for some $\delta_k > 0$.
- Previous results of this form holding for all values of D have been conditional on the GRH.

‘

Outlining our Goals

- We would like to find a bound of the form

$$NV_{d,k}(D) \gg D^{\delta_k}$$

  for some $\delta_k > 0$.
- Previous results of this form holding for all values of D have been conditional on the GRH.
- Our work has involved eliminating the GRH hypothesis. Doing so, we can no longer guarantee that our bound will hold for *all* values of D, but we can guarantee that it will be true "100 percent of the time"!

'

Definitions

- Let $\mathcal{S}_{d,k}$ be the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ satisfying our conditions on (d, k, D), plus some additional "local conditions".

Definitions

- Let $\mathcal{S}_{d,k}$ be the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ satisfying our conditions on (d, k, D), plus some additional "local conditions".
- Let $\mathcal{S}_{d,k}(X)$ be the subset of $\mathcal{S}_{d,k}$ such that $D \leq X$.

Definitions

- Let $\mathcal{S}_{d,k}$ be the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ satisfying our conditions on (d, k, D), plus some additional "local conditions".
- Let $\mathcal{S}_{d,k}(X)$ be the subset of $\mathcal{S}_{d,k}$ such that $D \leq X$.
- Let $\mathcal{S}_{d,k}^{NV}(X)$ be the subset of $\mathcal{S}_{d,k}(X)$ satisfying the bound

$$NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)}-\epsilon}.$$

Main Results

Theorem

We have the asymptotic formula

$$\#\mathcal{S}_{d,k}^{NV}(X) = \delta_{d,k} X + O_{d,k}(X^{1-\frac{1}{2(2k-1)}})$$

as $X \to \infty$*, for some explicit positive constant* $\delta_{d,k}$*.*

Main Results

Theorem

We have the asymptotic formula

$$\frac{\#\mathcal{S}_{d,k}^{NV}(X)}{\#\mathcal{S}_{d,k}(X)} = 1 + O(X^{-\frac{1}{2(2k-1)}})$$

as $X \to \infty$. *In particular, the bound*

$$NV_{d,k}(D) \gg_{\epsilon} D^{\frac{1}{2(2k-1)}-\epsilon}$$

holds for 100% of imaginary quadratic fields $K \in \mathcal{S}_{d,k}$.

Outline of Proof

Maria Ross

July 17, 2017

Galois Orbit

We define the Galois group $G_k = \mathrm{Gal}(\overline{\mathbb{Q}}/K(\zeta_{2k-1}))$, where ζ_{2k-1} denotes a primitive $2k-1^{st}$ root of unity.

Then G_k acts on the set of characters $\Psi_{d,k}(D)$ by

$$\psi \mapsto \psi^\sigma, \text{ where } \psi^\sigma = \sigma \circ \psi \text{ for } \sigma \in G_k,$$

and the Galois orbit of a character ψ is

$$\mathcal{O}_\psi = \{\psi^\sigma : \sigma \in G_k\}.$$

Strategy of Proof

- First, we find one character ψ so $L(\psi, k) \neq 0$. We do so by proving the following theorem:

Strategy of Proof

- First, we find one character ψ so $L(\psi, k) \neq 0$. We do so by proving the following theorem:

Theorem

If $D > 64d^4(k+1)^4$, there exists a $\psi \in \Psi_{d,k}(D)$ such that $L(\psi, k) \neq 0$.

Strategy of Proof

- Then, we use results of Shimura to show that

$$L(\psi, k) \neq 0 \iff L(\psi^\sigma, k) \neq 0$$

for all $\sigma \in G_k$.

Strategy of Proof

- Then, we use results of Shimura to show that

$$L(\psi, k) \neq 0 \quad \Longleftrightarrow \quad L(\psi^{\sigma}, k) \neq 0$$

for all $\sigma \in G_k$.

- It follows that $NV_{d,k}(D) \geq \#\mathcal{O}_{\psi}$.

Strategy of Proof

Let $\mathrm{Cl}_\ell(K)$ be the $\ell-$torsion subgroup of the class group $\mathrm{Cl}(K)$.

Strategy of Proof

Let $\mathrm{Cl}_\ell(K)$ be the $\ell-$torsion subgroup of the class group $\mathrm{Cl}(K)$.

- By Rohrlich, we have that under certain "local conditions",

$$\#\mathcal{O}_\psi = \frac{h(-D)}{|\mathrm{Cl}_{2k-1}(K)|}.$$

Strategy of Proof

Let $\mathrm{Cl}_\ell(K)$ be the $\ell-$torsion subgroup of the class group $\mathrm{Cl}(K)$.

- By Rohrlich, we have that under certain "local conditions",

$$\#\mathcal{O}_\psi = \frac{h(-D)}{|\mathrm{Cl}_{2k-1}(K)|}.$$

- Now we want to find a lower bound of the form

$$\frac{h(-D)}{|\mathrm{Cl}_{2k-1}(K)|} \gg D^{\delta_k}$$

for some $\delta_k > 0$.

Strategy of Proof

Recall that Siegel's Theorem gives us the bound

$$h(-D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}.$$

Strategy of Proof

Recall that Siegel's Theorem gives us the bound

$$h(-D) \gg_{\epsilon} D^{\frac{1}{2}-\epsilon}.$$

We want to find an upper bound of the form

$$|\mathrm{Cl}_{2k-1}(K)| \ll D^{\frac{1}{2}-\delta_k+\epsilon}.$$

Strategy of Proof

Recall that Siegel's Theorem gives us the bound

$$h(-D) \gg_\epsilon D^{\frac{1}{2}-\epsilon}.$$

We want to find an upper bound of the form

$$|\mathrm{Cl}_{2k-1}(K)| \ll D^{\frac{1}{2}-\delta_k+\epsilon}.$$

Combining such a bound with Siegel's theorem would give

$$NV_{d,k}(D) \geq \#\mathcal{O}_\psi \gg D^{\delta_k-\epsilon}.$$

Bounding ℓ-torsion in Class Groups

Theorem (Ellenberg and Venkatesh, 2005)

Assuming GRH,

$$|Cl_\ell(K)| \ll_\epsilon D^{\frac{1}{2}-\frac{1}{2\ell}+\epsilon}.$$

Theorem (Ellenberg, Pierce, Wood (2016))

The bound

$$|Cl_\ell(K)| \ll_\epsilon D^{\frac{1}{2}-\frac{1}{2\ell}+\epsilon}$$

holds unconditionally for all imaginary quadratic fields K *with* $D \leq X$ *except an "exceptional set" of size* $O(X^{1-\frac{1}{2\ell}})$.

Bounding the ℓ-torsion subgroup

A restatement of the results of Ellenberg, Pierce, and Wood (2016) yields

$$\frac{\#\{K : D \leq X, |\mathrm{Cl}_\ell(K)| \ll_\epsilon D^{\frac{1}{2}-\frac{1}{2\ell}+\epsilon}\}}{\#\{K : D \leq X\}} = 1 + O(X^{-\frac{1}{2\ell}}).$$

Under our particular conditions...

Recall that $\mathcal{S}_{d,k}$ is the set of all imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ that satisfy our conditions on (d, k, D), along with some "local conditions".

We incorporate our local conditions into the work of Ellenberg, Pierce, and Wood to get an asymptotic formula for the number of imaginary quadratic fields K with $D \leq X$ that satisfy our conditions:

$$\#\mathcal{S}_{d,k}(X) = \delta_{d,k}X + O(X^{\frac{1}{2}})$$

for an explicit constant $\delta_{d,k}$.

Let $\mathcal{S}_{d,k}^{Tor}$ denote the subset of $\mathcal{S}_{d,k}$ such that the torsion bound is satisfied, i.e., $|\mathrm{Cl}_\ell(K)| \ll_\epsilon D^{\frac{1}{2}-\frac{1}{2\ell}+\epsilon}$.

We prove that if K is in the set $\mathcal{S}_{d,k}^{Tor}$, then

$$NV_{d,k}(D) \ \geq \#\mathcal{O}_\psi \gg D^{\frac{1}{2(2k-1)}-\epsilon}.$$

Thus, $\mathcal{S}_{d,k}^{Tor}$ is a subset of $\mathcal{S}_{d,k}^{NV}$, the set of fields in $\mathcal{S}_{d,k}$ with

$$NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)}-\epsilon}.$$

Finding an Asymptotic Formula

We can decompose $\mathcal{S}_{d,k}(X)$ into the disjoint union of $\mathcal{S}_{d,k}^{NV}(X)$ and its complement, $\mathcal{S}_{d,k}^{-}(X)$. Then,

$$\#\mathcal{S}_{d,k}^{NV}(X) = \#\mathcal{S}_{d,k}(X) - \#\mathcal{S}_{d,k}^{-}(X).$$

From Ellenberg, Pierce, and Wood, we know that the number of fields with our particular conditions not satisfying the torsion bound is bounded above by $O(X^{1-\frac{1}{2(2k-1)}})$.

So, we can use $O(X^{1-\frac{1}{2(2k-1)}})$ as an upper bound for $\#\mathcal{S}_{d,k}^{-}(X)$.

Finding an Asymptotic Formula

Then, we combine our asymptotic formula for $\#\mathcal{S}_{d,k}(X)$ with this upper bound on the number of fields that don't satisfy $NV_{d,k}(D) \gg_\epsilon D^{\frac{1}{2(2k-1)}-\epsilon}$ to get

$$\#\mathcal{S}_{d,k}^{NV}(X) = \delta_{d,k}X + O_{d,k}(X^{1-\frac{1}{2(2k-1)}})$$

for explicit positive constant $\delta_{d,k}$.

Finally, we consider the ratio of $\#\mathcal{S}_{d,k}^{NV}(X)$ to $\#\mathcal{S}_{d,k}(X)$ and arrive at our density statement.

Acknowledgements

We would like to thank

- Our advisor, Dr. Riad Masri,
- Our graduate student mentor, Wei-Lun Tsai,
- Texas A&M University for hosting,
- The NSF for funding.

Thank you!

J. Ellenberg and A. Venkatesh, *Reflection Principles and Bounds for Class Group Torsion*, Int. Math. Res. Not. no.1, Art ID rnm002 (2007).

J. Ellenberg, L. B. Pierce, M. Matchett Wood, *On ℓ-torsion in Class Groups of Number Fields*, arXiv:1606.06103 [math.NT].

G. Shimura, *The special values of the zeta functions associated to cusp forms*, Comm. Pure Appl. Math. 29 (1976), 783-804.