Zeros of Newform Eisenstein Series on $\Gamma_0(N)$

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Definitions

$SL_2(\mathbb{Z})$

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$\Gamma_0(N)$

A subgroup of $SL_2(\mathbb{Z})$ is $\Gamma_0(N)$, defined as

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

Modular Forms

Definition: *Modular Form*

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- and $\lim_{z \to i\infty} f(z)$ exists.

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Consider the weight k Eisenstein series, $E_k : \mathbb{H} \to \mathbb{C}$, defined as

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^k},$$

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- Rankin and Swinnerton-Dyer studied the zeros of $E_k(z)$.
- The zeros of $E_k(z)$ rest on the the boundary of the fundamental domain, $\mathcal{F}$, where

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

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Consider the weight k Newform Eisenstein series, $E_{\chi_1,\chi_2,k} : \mathbb{H} \to \mathbb{C}$, on the congruence subgroup $\Gamma_0(q_1 q_2)$ defined as

$$E_{\chi_1,\chi_2,k}(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{\chi_1(c)\chi_2(d)}{(cq_2 z + d)^k},$$

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- We wish to find zeros of $E_{\chi_1,\chi_2,k}(z)$ as weight k is sufficiently large.
- We utilize two different expansions to locate the zeros.

The Fourier Expansion for $E_{\chi_1,\chi_2,k}(z)$ is defined as

$$E_{\chi_1,\chi_2,k}(z) = e(\chi_1, \chi_2, k)\sum_{n=1}^{\infty}\left(\sum_{ab=n}\chi_1(a)\overline{\chi_2}(b)b^{k-1}\right)e(nz),$$

where

- $e(nz) = e^{2\pi inz}$
- $e(\chi_1, \chi_2, k)$ is some constant independent of z.

Fourier Expansion

Simplification

$$\text{Let } F(z) = \sum_{n=1}^{\infty} \left(\sum_{ab=n} \chi_1(a)\overline{\chi_2}(b) b^{k-1} \right) e(nz).$$

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where

$$|\delta(z)| \leq \sum_{n=1}^{\infty} n^{k-1} \exp(-2\pi ny) \left(\sum_{\substack{b|n \\ b<n}} \left(\frac{b}{n}^{k-1} \right) \right).$$

Rouché's Theorem

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Let F and h be two complex-valued functions which are complex analytic on a closed region V with rectangular boundary ∂V. If

$$|F(z) - h(z)| < |F(z)| + |h(z)|,$$

for all $z \in \partial V$, then F and h have the same number of zeros, including multiplicity, in V.

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Why Rouché's Theorem?:

- We count the zeros of a good approximation to F, namely h.
- We consequently know the number of zeros of the original function F.

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- They looked at Hecke cusp forms for $\text{Im}(z) \gg \sqrt{k}$.

Fourier Expansion
Approximation

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- The functions are best approximated by one term of the Fourier expansion at $n = \ell$ with $y = \frac{k-1}{2\pi\ell}$.

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For our purposes with Newform Eisenstein Series:

- Fourier expansion is used to approximate $E_{\chi_1,\chi_2,z}(z)$ when $\text{Im}(z) \gg \sqrt{k}$.

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For our purposes with Newform Eisenstein Series:

- Fourier expansion is used to approximate $E_{\chi_1,\chi_2,z}(z)$ when $\mathrm{Im}(z) \gg \sqrt{k}$.
- The $n = \ell$ and $n = \ell + 1$ terms of the Fourier expansion gives a good approximation for $E_{\chi_1,\chi_2,k}(z)$ for $y = \mathrm{Im}(z)$ in the range:

$$\frac{k-1}{2\pi(\ell+1)} = y_{\ell+1} \leq y \leq y_\ell = \frac{k-1}{2\pi\ell}.$$

Main Term, $h_\ell(z)$

Lemma 1

Consider the $n = \ell$ and $n = \ell + 1$ terms of the Fourier expansion:

$$\begin{aligned} h_\ell(z) &= \overline{\chi_2}(\ell)\ell^{k-1}e(\ell z) + \overline{\chi_2}(\ell+1)(\ell+1)^{k-1}e((\ell+1)z) \\ &= f_\ell(z) + f_{\ell+1}(z). \end{aligned}$$

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Lemma (1)

The main term $h_\ell(z)$ has a unique zero $x_0 + iy_0$ in the region $-\frac{1}{2} < x \leq \frac{1}{2}$ and $y_{\ell+1} \leq y \leq y_\ell$, with x_0 and y_0 given as

$$e(x_0) = -\overline{\chi_2}(\ell)\chi_2(\ell+1)$$

and

$$y_0 = \frac{k-1}{2\pi}\left|\log\left(1 - \frac{1}{\ell+1}\right)\right|.$$

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$$\beta(z) = f_{\ell+2}(z) + f_{\ell-1}(z) + \varepsilon_1(z) + \varepsilon_2(z) + \delta(z)$$

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- $\varepsilon_1(z) = \displaystyle\sum_{n=1}^{\ell-2} f_n(z)$ and $\varepsilon_2(z) = \displaystyle\sum_{n=\ell+3}^{\infty} f_n(z)$

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- $\delta(z)$ as previously defined.

Main Theorem

Define a natural normalization factor of $F(z)$ as

$$N(y,k) = \frac{(2\pi y)^k}{\Gamma(k)},$$

and define the region V_ℓ as

$$V_\ell = \left\{ z \in \mathbb{H} : x_0 - \frac{1}{2} \leq x \leq x_0 + \frac{1}{2}, y_{\ell+1} \leq y \leq y_\ell \right\}.$$

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Theorem

Let ℓ be a natural number with $(\ell, q_2) = (\ell+1, q_2) = 1$ and $\ell \leq \epsilon\sqrt{k}$ for a small $\epsilon > 0$. Then, $E_{\chi_1,\chi_2,k}(z)$ has exactly one zero in V_ℓ.

Method for Proof

Rouché's Theorem

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Let F and h be two complex-valued functions which are complex analytic on a closed region V with rectangular boundary ∂V. If

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Then, on ∂V_ℓ, it suffices to show:

$$N(y,k)\,|\beta(z)| < N(y,k)\,|h_\ell(z)|.$$

Then, $F(z)$ will have exactly one zero in V_ℓ.

Lemma (2)

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- $y = y_\ell$, the top boundary;
- $y = y_{\ell+1}$, the bottom boundary;
- $x = x_0 \pm \frac{1}{2}$, the left and right boundaries.

Lemma (3)

For all $z \in V_\ell$,

$$N(y,k)\,|\beta(z)| \ll \frac{\sqrt{k}}{2^k \ell} + \frac{\sqrt{k}}{\ell} \exp\left(-\frac{k}{4\ell^2}\right).$$

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- $f_{\ell+2}(z)$ and $f_{\ell-1}(z)$;
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- $\delta(z)$.

Proof of Theorem

From Lemma [1], [2], and [3], the theorem

Theorem

The function $E_{\chi_1,\chi_2,k}(z)$ has exactly one zero for in the region V_ℓ.

is proven as

$$\frac{\sqrt{k}}{2^k\ell} + \frac{\sqrt{k}}{\ell}\exp\left(-\frac{k}{4\ell^2}\right) < \frac{\sqrt{k}}{\ell}.$$

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