Effective Bounds for Traces of Singular Moduli

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Thank you

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Outline

The upper half plane and modular group

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The upper half plane and modular group

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- Let $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$
- $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by linear fractional transformations: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, then the group action is defined by
$$\gamma(z) = \frac{az + b}{cz + d}.$$

The J-function

- The classical modular j-function is defined as

$$j(z) := e(-z) + 744 + \sum_{n>0} a(n)e(nz), \quad z \in \mathbb{H}$$

where $e(z) := e^{2\pi iz}$ and $a(n) \in \mathbb{Z}$ is a Fourier coefficient for which an explicit formula can be found.

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- Define $J(z) := j(z) - 744$.

- Note that

$$J(\gamma z) = J(z)$$

  for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, and so J is an automorphic function.

Binary Quadratic Forms

- Let $\mathcal{Q}_d$ be the set of primitive, positive-definite, integral, binary quadratic forms

$$Q(x, y) = [a_Q, b_Q, c_Q] = a_Q x^2 + b_Q xy + c_Q y^2$$

with discriminant $d = b_Q^2 - 4a_Q c_Q < 0$.

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- There is a right action of $\mathrm{SL}_2(\mathbb{Z})$ on Q_d given by

$$Q \circ M(x, y) = Q(\alpha x + \beta y, \gamma x + \delta y)$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

The Class Number

- The quotient $Q_d/\mathrm{SL}_2(\mathbb{Z})$ is finite. Let

$$h(d) := |Q_d/\mathrm{SL}_2(\mathbb{Z})|$$

be the *class number* of d.

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Theorem (Siegel)

For all $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$h(d) \geq C(\epsilon)\,|d|^{\frac{1}{2}+\epsilon}.$$

Singular Moduli

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- A *CM point* is the root of $Q(x,1)$ in $\mathbb{H}$ given by

$$\tau_Q = \frac{-b_Q + i\sqrt{|d|}}{2a_Q}.$$

- The values $J(\tau_Q)$ are algebraic numbers called *singular moduli*.

Traces of singular moduli

- We define the trace of singular moduli by:

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- The trace is well defined because if $[Q_1] = [Q_2]$, then $\gamma\tau_{Q_1} = \tau_{Q_2}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and J is automorphic.

Zagier's generating function

- Let

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$$g_{Zag}(z) := e(-z\,|d|) + \sum_{d \equiv 0,1 \pmod 4} Tr_d(J) e(z\,|d|).$$

- A remarkable theorem of Zagier asserts that $g_{Zag}(z)$ is a weakly holomorphic modular form of weight $3/2$ for $\Gamma_0(4)$.

Importance

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- A problem of central importance in number theory is to bound Fourier coefficients of modular forms.
- As a consequence of our main theorem, we will give effective bounds for the Fourier coefficients of $g_{Zag}(z)$.

A Theorem of Duke

Theorem (Duke, 2006)

There is an absolute constant $\delta > 0$ such that

$$Tr_d(J) = \sum_{\substack{[Q] \in Q_d / SL_2(\mathbb{Z}) \\ \mathrm{Im}(\tau_Q) > 1}} e(-\tau_Q) - 24h(d) + \mathcal{O}(|d|^{\frac{1}{2}-\delta}).$$

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- Note that $\frac{\mathcal{O}(|d|^{\frac{1}{2}-\delta})}{h(d)} \to 0$ as $|d| \to \infty$ by Siegel's Theorem.

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- Thus Duke's theorem implies that

$$\frac{Tr_d(J) - \sum_{\substack{[Q]\in Q_d/\mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}(\tau_Q)>1}} e(-\tau_Q)}{h(d)} \to -24$$

as $|d| \to \infty$. This confirmed a conjecture of Bruinier, Jenkins, and Ono.

Special case of our Main Theorem

Theorem

$$Tr_d(J) = \sum_{\substack{[Q] \in Q_d / SL_2(\mathbb{Z}) \\ \mathrm{Im}(\tau_Q) > 1}} e(-\tau_Q) - 24h(d) + E(d)$$

where

$$|E(d)| \leq (1.72 \times 10^6) h(d).$$

A corollary

Corollary

$$|Tr_d(J)| \leq e^{\pi\sqrt{|d|}}(1.72 \times 10^6)h(d)$$

Comparison with Duke's Theorem

- Duke proved that

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  converges by saving a power of d in the error term over the "trivial" bound $h(d) \ll \log(|d|)\sqrt{|d|}$.

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- Therefore we require a new method for our main theorem.

Reduced forms

- The *fundamental domain* for $SL_2(\mathbb{Z})$ acting on $\mathbb{H}$ is the region

$$\mathcal{F} := \left\{ z \in \mathbb{C} \,\middle|\, |z| > 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{C} \,\middle|\, -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0, |z| = 1 \right\}.$$

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- A form Q is said to be *reduced* if its CM point lies in $\mathcal{F}$.
- Each $[Q] \in Q_d/SL_2(\mathbb{Z})$ contains a unique reduced form.

Summing over reduced forms

- Let $Q_1, \ldots, Q_{h(d)}$ be the set of reduced forms representing the equivalence classes in $\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$.

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- Let $Q_1, \ldots, Q_{h(d)}$ be the set of reduced forms representing the equivalence classes in $Q_d/\mathrm{SL}_2(\mathbb{Z})$.

- We can sum over $Q_1, \ldots, Q_{h(d)}$ in the trace of $J(z)$:

$$Tr_d(J) = \sum_{i=1}^{h(d)} J(\tau_{Q_i}).$$

The Poincaré series

- For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $z \in \mathbb{H}$, define the *Maass-Poincaré series*

$$F(z, s) := 2\pi \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi \mathrm{Im}(\gamma z)) e(-\mathrm{Re}(\gamma z))$$

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- And

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z}^+ \cup \{0\} \right\}$$

is the subset of $\mathrm{SL}_2(\mathbb{Z})$ that stabilizes the cusp at infinity.

Proposition

Proposition

- *The limit*

$$\lim_{s \to 1^+} F(z,s)$$

exists and is given by

$$F(z,1) = e(-z) + \sum_{n=0}^{\infty} b(n) e(nz)$$

where $b(0) = 24$ *and*

$$b(n) = 2\pi n^{-\frac{1}{2}} \sum_{c>0} \frac{S(n,-1;c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c}\right), \qquad n > 0.$$

- $J(z) = F(z,1) - 24.$

The Kloosterman sum

- $S(a, b; c)$ is the ordinary *Kloosterman sum*

$$S(a, b; c) := \sum_{\substack{d \pmod{c} \\ (c,d)=1}} e\left(\frac{a\overline{d} + bd}{c}\right)$$

where $\overline{d}$ is the multiplicative inverse of $d \pmod{c}$.

The Fourier expansion

$F(z,s)$ has a Fourier expansion given by

$$F(z,s) = 2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e(-x) + c_s y^{1-s} + 4\pi \sum_{n\neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

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and

$$b(n;s) := \sum_{c>0} \frac{S(n,-1;c)}{c} \begin{cases} I_{2s-1}\left(\frac{4\pi\sqrt{n}}{c}\right) & n > 0 \\ J_{2s-1}\left(\frac{4\pi\sqrt{|n|}}{c}\right) & n < 0. \end{cases}$$

The first two terms

$$F(z,s) = \mathbf{2\pi y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y) e(-x) + c_s y^{1-s}} + 4\pi \sum_{n\neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

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- These are analytic functions on $\mathbb{C}$.
- We want to show that for $z \in \mathbb{H}$, the sum

$$B(z,s) := \sum_{n\neq 0} b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx)$$

converges absolutely for all $s \in \mathbb{R}$ such that $s \geq 1$.

Bounding the Fourier coefficients

Proposition

For $s \in \mathbb{R}$ such that $s \geq 1$,

$$|b(n;s)| \leq \begin{cases} C_1(s)\,|n|^s & n < 0 \\ C_2(s) n^s e^{4\pi\sqrt{n}} & n > 0 \end{cases}$$

and

$$\left|K_{s-\frac{1}{2}}(2\pi\,|n|\,y)\right| \leq C_3(s)\frac{e^{-2\pi|n|y}}{\sqrt{|n|\,y}}$$

where C_1, C_2, and C_3 are explicit constants that depend on s.

Key ideas

- The Weil bound:

$$|S(a, b; c)| \leq \tau(c)(a, b, c)^{1/2} c^{1/2}$$

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  where τ is the divisor function.
- A careful study of the asymptotics of the I, J, and K Bessel functions.

Bounding the infinite sum

Using these bounds we can show that for $z \in \mathbb{H}$,

$$|B(z,s)| \leq \sum_{n \neq 0} \left| b(n;s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx) \right| < \infty$$

for all $s \in \mathbb{R}$ such that $s \geq 1$.

The Fourier expansion of $F(z,1)$

Thus after some manipulation we find that

$$\lim_{s\to 1^+} F(z,s) = F(z,1) = e(-z) + 24 - e(-\overline{z}) + 2\pi \sum_{n<0} b(n;1)\,|n|^{-\frac{1}{2}}\, e(n\overline{z}) + 2\pi \sum_{n>0} b(n;1) n^{-\frac{1}{2}} e(nz).$$

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- Note that $F(z, 1)$ and $J(z)$ have the same principal part.

- Hence the function $\phi(z)$ is bounded on $\mathbb{H}$.

The hyperbolic Laplacian operator

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- **Fact:** If f is a holomorphic function on $\mathbb{H}$ then $\Delta f(z) = 0$.

- Since $J(z)$ is holomorphic on $\mathbb{H}$, $\Delta J(z) = 0$.

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- So $\Delta F(z,1) = 0$.
- Therefore $\Delta\phi(z) = 0$, so $\phi(z)$ is harmonic.

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- Since

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  we have that

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$$\phi(z) = F(z,1) - J(z) = 24$$

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  and thus

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- This proves the second part of the proposition.

The anti-holomorphic part

Recall:

$$\begin{aligned} F(z,1) = e(-z) + 24 &- \boldsymbol{e(-\overline{z})} \\ &+ \boldsymbol{2\pi \sum_{n<0} b(n;1)\,|n|^{-\frac{1}{2}}\, e(n\overline{z})} \\ &+ 2\pi \sum_{n>0} b(n;1) n^{-\frac{1}{2}} e(nz). \end{aligned}$$

The anti-holomorphic part (cont.)

- Since $F(z,1) - 24 = J(z)$ and $J(z)$ is holomorphic, the anti-holomorphic part of $F(z,1)$ is zero, hence

$$F(z,1) = e(-z) + 24 + 2\pi \sum_{n>0} b(n;1) n^{-\frac{1}{2}} e(nz)$$

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$$F(z,1) = e(-z) + 24 + 2\pi \sum_{n>0} b(n;1) n^{-\frac{1}{2}} e(nz)$$

- We can conclude that $b(0) = 24$ and

$$\begin{aligned} b(n) &= 2\pi b(n;1) n^{-\frac{1}{2}} \\ &= 2\pi n^{-\frac{1}{2}} \sum_{c>0} \frac{S(n,-1;c)}{c} I_1\left(\frac{4\pi\sqrt{n}}{c}\right), \qquad n > 0. \end{aligned}$$

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where

$$E(d) := \sum_{n=0}^{\infty} b(n) \sum_{i=1}^{h(d)} e(n\tau_{Q_i}).$$

The main term

- We can write

$$\sum_{i=1}^{h(d)} e(-\tau_{Q_i}) = \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})>1}} e(-\tau_{Q_i}) + \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} e(-\tau_{Q_i}).$$

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- Note that

$$\left| \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} e(-\tau_{Q_i}) \right| \leq \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} |e(-\tau_{Q_i})|$$

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$$\sum_{i=1}^{h(d)} e(-\tau_{Q_i}) = \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})>1}} e(-\tau_{Q_i}) + \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} e(-\tau_{Q_i}).$$

- Note that

$$\left| \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} e(-\tau_{Q_i}) \right| \leq \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} |e(-\tau_{Q_i})|$$

$$= \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} \left| e^{-2\pi i \mathrm{Re}(\tau_{Q_i})} e^{2\pi \mathrm{Im}(\tau_{Q_i})} \right|$$

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Bounding $|E(d)|$

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$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})| .$$

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Bounding $|E(d)|$ (cont.)

- Since $\tau_{Q_1}, \ldots, \tau_{Q_{h(d)}}$ lie in the fundamental domain $\mathcal{F}$,

$$\mathrm{Im}(\tau_{Q_i}) \geq \frac{\sqrt{3}}{2}$$

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- Thus

$$\sum_{i=1}^{h(d)} e^{-2\pi n \mathrm{Im}(\tau_{Q_i})} \leq h(d) e^{-\pi n \sqrt{3}}. \qquad (1)$$

Bounding $|E(d)|$ (cont.)

- **Recall:** For $s \in \mathbb{R}$ such that $s \geq 1$,

$$|b(n;s)| \leq \begin{cases} C_1(s)\,|n|^s & n < 0 \\ C_2(s)n^s e^{4\pi\sqrt{n}} & n > 0. \end{cases}$$

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- So, setting $s = 1$,

$$|b(n;1)| \leq (105.20) n e^{4\pi\sqrt{n}}, \qquad n > 0. \qquad (2)$$

Bounding $|E(d)|$ (cont.)

- Combining (1) and (2), we get

$$|E(d)| \leq \sum_{n=0}^{\infty} |b(n)| \sum_{i=1}^{h(d)} |e(n\tau_{Q_i})| \leq (1.72 \times 10^6) h(d).$$

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- Combined with our earlier observation that

$$\sum_{i=1}^{h(d)} e(-\tau_{Q_i}) = \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})>1}} e(-\tau_{Q_i}) + \sum_{\substack{Q_i \\ \mathrm{Im}(\tau_{Q_i})\leq 1}} e(-\tau_{Q_i})$$

this completes the proof of the theorem.

Recap

Theorem

$$Tr_d(J) = \sum_{\substack{[Q]\in Q_d/SL_2(\mathbb{Z}) \\ \mathrm{Im}(\tau_Q)>1}} e(-\tau_Q) - 24h(d) + E(d)$$

where

$$|E(d)| \leq (1.72 \times 10^6)h(d).$$