Geometry of $\mathbb{R}$ Roots of 9×9 Polynomial Systems

Luis Feliciano
Texas A&M University

July 16, 2018
REU: DMS – 1757872

Motivation

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

$$f_2(x_8, x_9) = c_6 x_9^2 + c_7 x_8 x_9 + c_8 x_8 + c_9 x_9 + c_{10}$$

A Quadratic Pentanomial 2x2 system!

Motivation

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

$$f_2(x_8, x_9) = c_6 x_9^2 + c_7 x_8 x_9 + c_8 x_8 + c_9 x_9 + c_{10}$$

A Quadratic Pentanomial 2x2 system!

Solving polynomial equations becomes more and more complicated as we increase the number of terms and variables.

Motivation

$$f_1(x_8, x_9) = c_1x_8^2 + c_2x_8x_9 + c_3x_8 + c_4x_9 + c_5$$

$$f_2(x_8, x_9) = c_6x_9^2 + c_7x_8x_9 + c_8x_8 + c_9x_9 + c_{10}$$

A Quadratic Pentanomial 2x2 system!

Solving polynomial equations becomes more and more complicated as we increase the number of terms and variables.

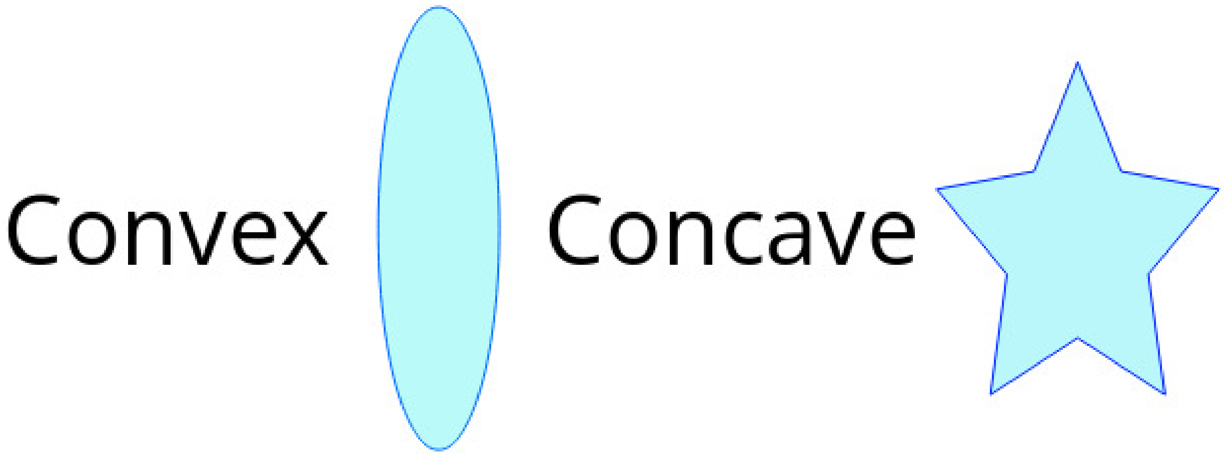
Today we'll take a look at some constructions that give us rough approximations for roots in a fraction of the time!

A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set

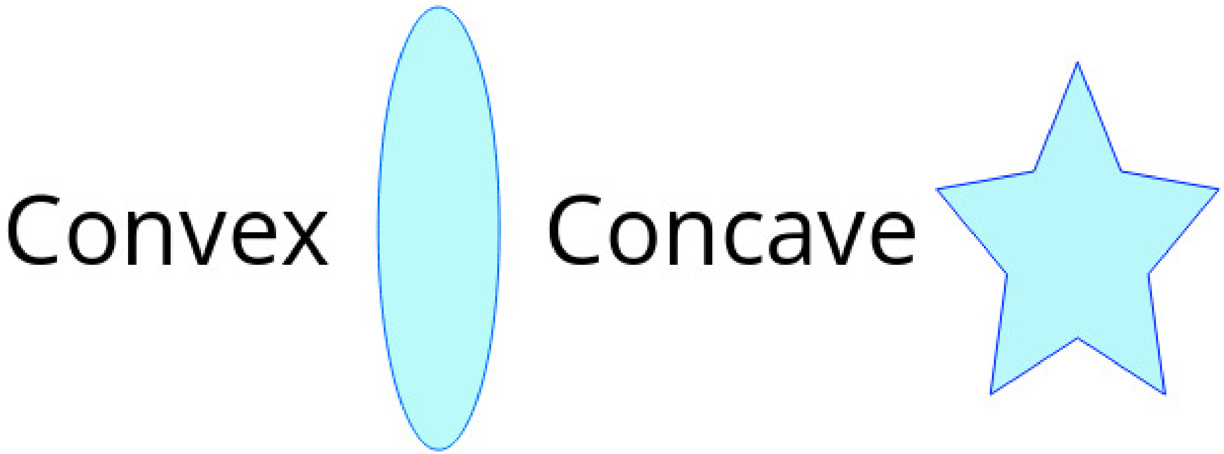
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



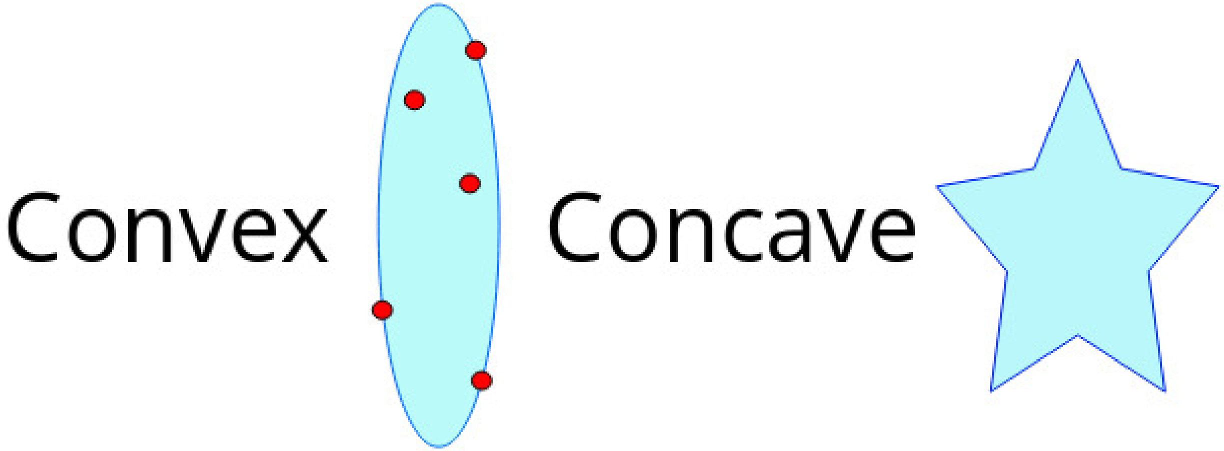
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



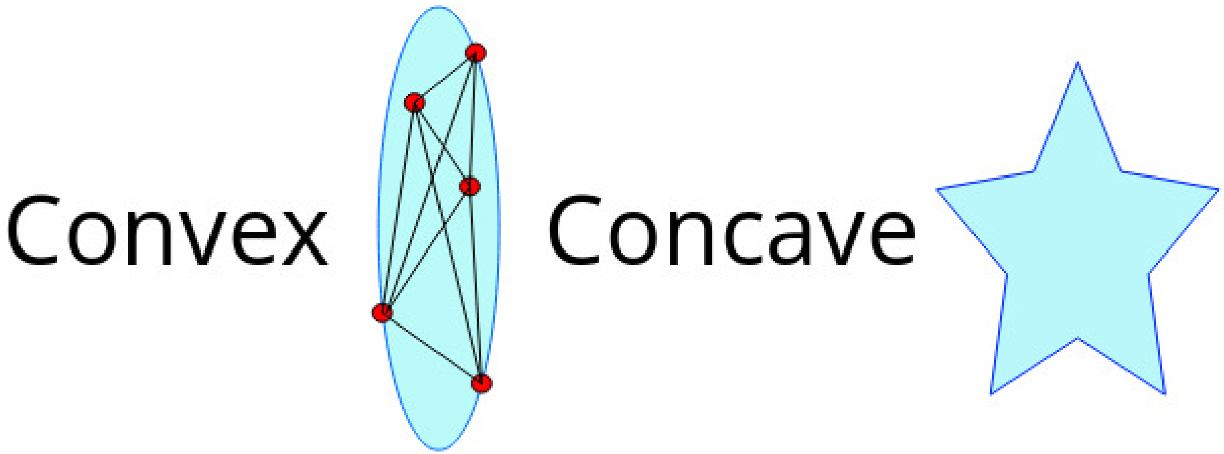
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



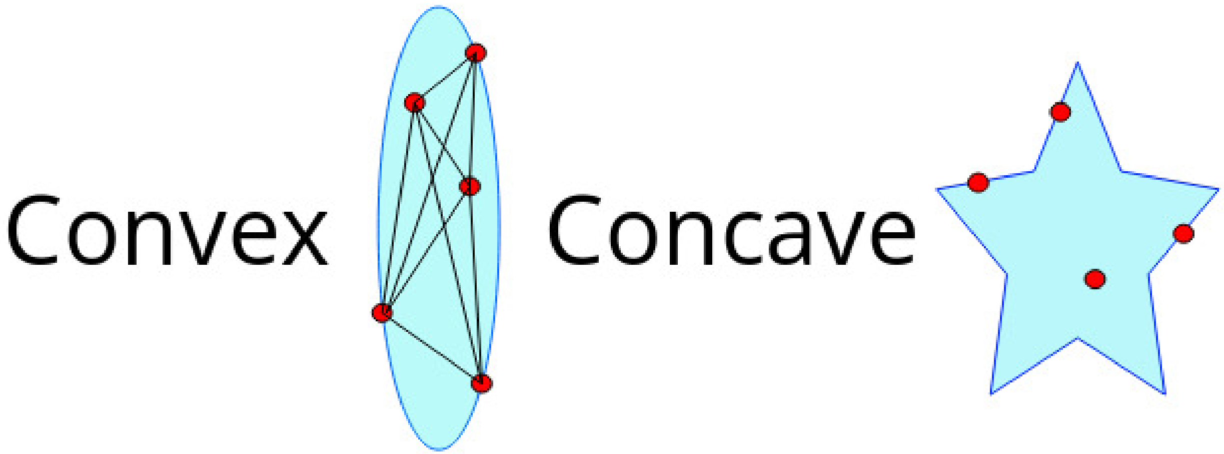
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



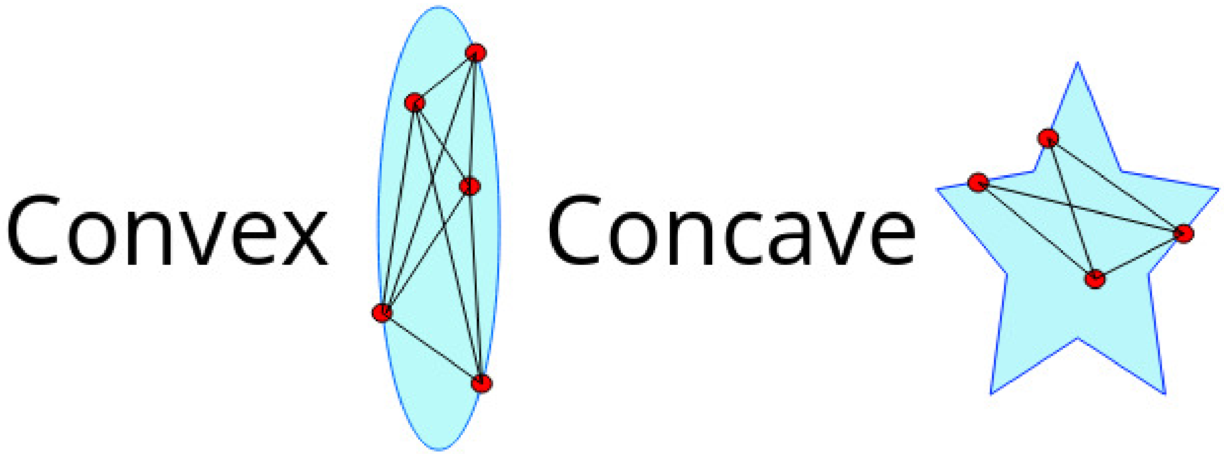
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



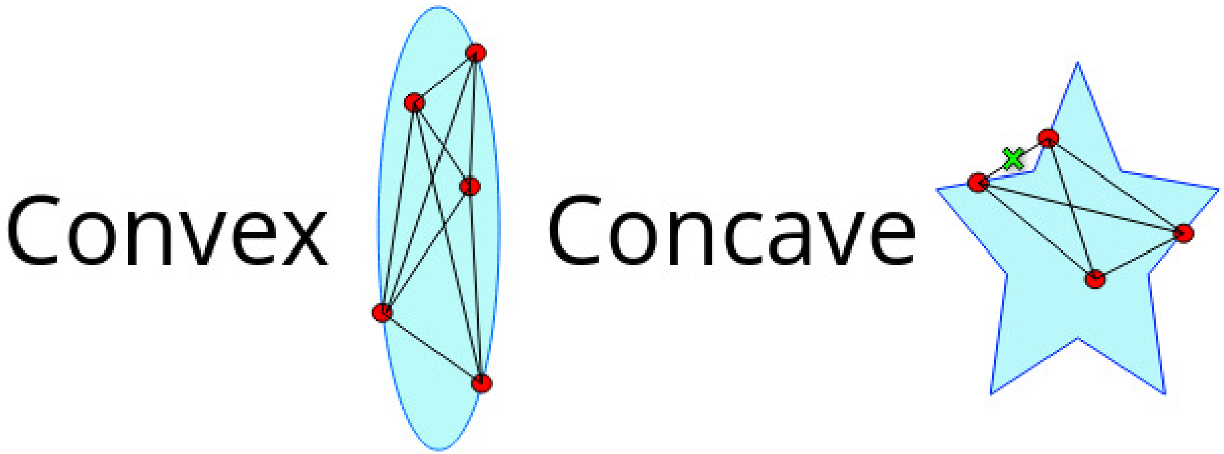
A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



A Quick Review

A *convex set* is a set of points such that, given any two points P, Q, then the line segment PQ is also in the set



Convex Hull

Denoted by $\text{conv}\{\cdot\}$

Convex Hull

Denoted by $\text{conv}\{\cdot\}$

For any $S \subset \mathbb{R}^n$

Convex Hull

Denoted by conv$\{\cdot\}$

For any $S \subset \mathbb{R}^n$

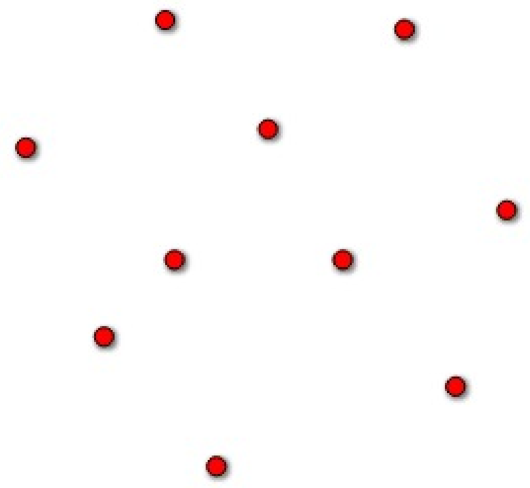
conv$\{S\}$:= smallest convex set containing S

Convex Hull

Denoted by $\text{conv}\{\cdot\}$

For any $S \subset \mathbb{R}^n$

$\text{conv}\{S\} :=$ smallest convex set containing S

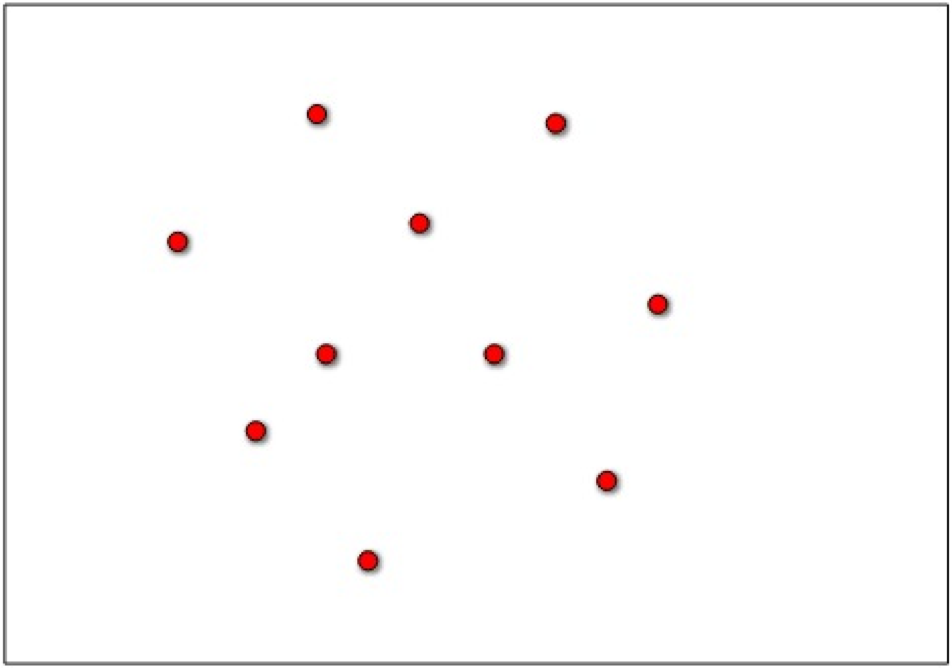


Convex Hull

Denoted by $\text{conv}\{\cdot\}$

For any $S \subset \mathbb{R}^n$

$\text{conv}\{S\} :=$ smallest convex set containing S

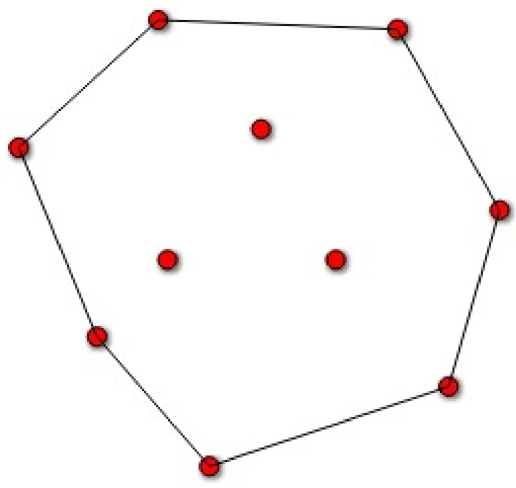


Convex Hull

Denoted by $\text{conv}\{\cdot\}$

For any $S \subset \mathbb{R}^n$

$\text{conv}\{S\} :=$ *smallest* convex set containing S



Newton Polytope

Denoted by $\mathrm{Newt}(\cdot)$

Newton Polytope

Denoted by Newt$(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

Newton Polytope

Denoted by $\text{Newt}(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

Newton Polytope

Denoted by $\text{Newt}(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$$

Newton Polytope

Denoted by Newt$(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$$

Ex:

$$f(x,y) = 1 + x^2 + y^3 - 100xy$$

Newton Polytope

Denoted by Newt$(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$$

<u>**Ex**</u>:

$$\begin{aligned} f(x,y) &= 1 + x^2 + y^3 - 100xy \\ &= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1 \end{aligned}$$

Newton Polytope

Denoted by Newt$(\cdot)$

$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$

Ex:

$$\begin{aligned} f(x,y) &= 1 + x^2 + y^3 - 100xy \\ &= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1 \end{aligned}$$

(0,0)
(2,0)
(0,3)
(1,1)

Newton Polytope

Denoted by Newt$(\cdot)$

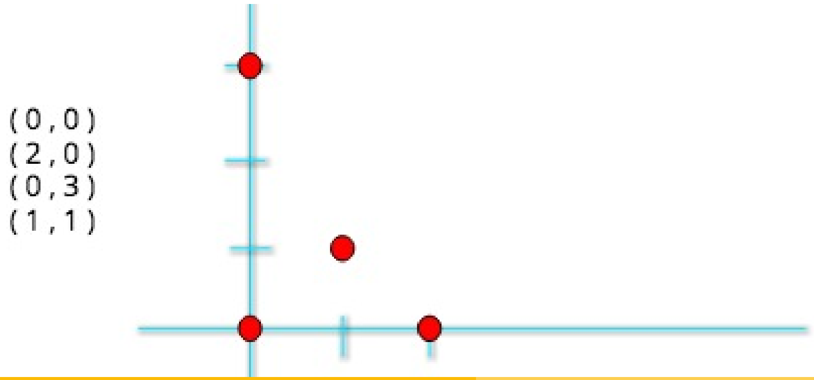
$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$

Ex:

$$\begin{aligned} f(x,y) &= 1 + x^2 + y^3 - 100xy \\ &= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1 \end{aligned}$$



Newton Polytope

Denoted by $\text{Newt}(\cdot)$

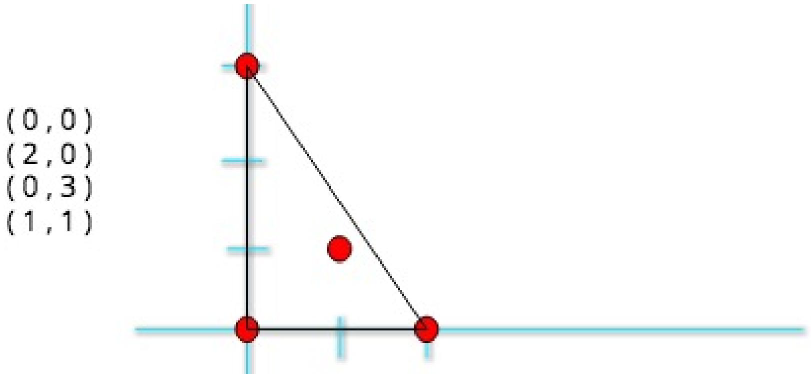
$$f(x) = \sum_{i=1}^{t} c_i x^{a_i}$$

where $x^{a_i} = x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}$

$$\text{Newt}(f) := \text{conv}\{a_i \mid c_i \neq 0\}$$

Ex:

$$\begin{aligned} f(x,y) &= 1 + x^2 + y^3 - 100xy \\ &= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1 \end{aligned}$$

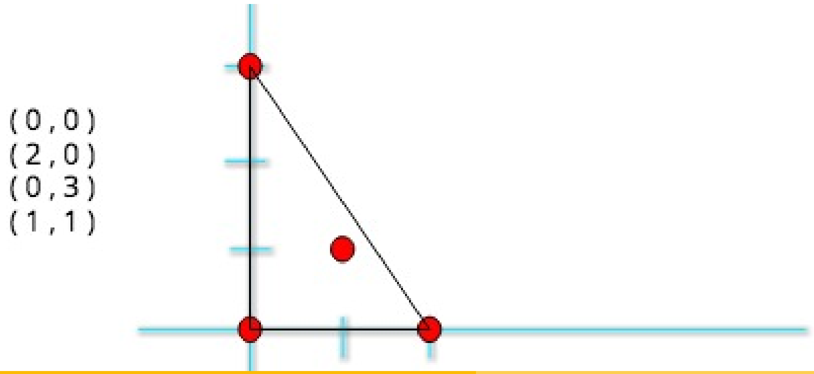


Newton Polytope

Although we didn't make use of the following in our research...

<u>Ex</u>:

$$f(x,y) = 1 + x^2 + y^3 - 100xy$$
$$= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$$



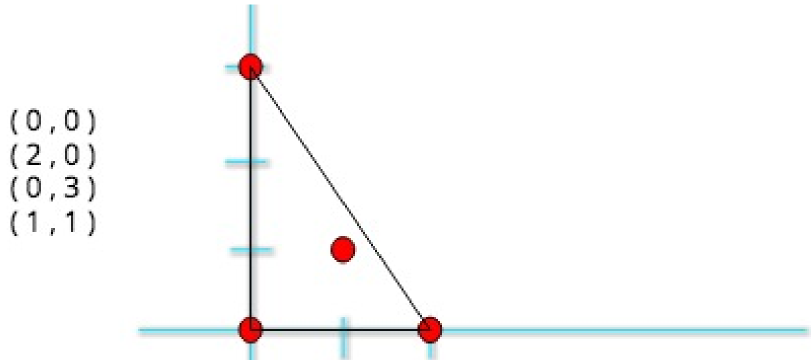
Newton Polytope

Although we didn't make use of the following in our research...

The volume of the Newton polytope can be used to compute the degree of the corresponding hypersurface, and via mixed volumes, the number of roots of systems of equations!

<u>**Ex**</u>:

$$f(x,y) = 1 + x^2 + y^3 - 100xy$$
$$= 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$$



Archimedean Newton Polytope

Archimedean Newton Polytope

Denoted by $\mathrm{ArchNewt}(f)$

Archimedean Newton Polytope

Denoted by ArchNewt(f)

$\mathrm{ArchNewt}(f) := \mathrm{conv}\{(a_i, -\mathrm{Log}|c_i|) \mid i \in \{1, \ldots, t\}, c_i \neq 0\}$

Archimedean Newton Polytope

Denoted by $\text{ArchNewt}(f)$

$$\text{ArchNewt}(f) := \text{conv}\{(a_i, -\text{Log}|c_i|) \mid i \in \{1, \ldots, t\}, c_i \neq 0\}$$

$$f(x, y) = 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$$

Archimedean Newton Polytope

Denoted by ArchNewt(f)

$\text{ArchNewt}(f) := \text{conv}\{(a_i, -\text{Log}|c_i|) \mid i \in \{1, \ldots, t\}, c_i \neq 0\}$

$f(x, y) = 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$

$\Rightarrow \text{ArchNewt}(f) =$

$\text{conv}\{(0, 0, -\text{Log}(1))$

Archimedean Newton Polytope

Denoted by ArchNewt(f)

$\text{ArchNewt}(f) := \text{conv}\{(a_i, -\text{Log}|c_i|) \mid i \in \{1, \ldots, t\}, c_i \neq 0\}$

$f(x, y) = 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$

$\Rightarrow \text{ArchNewt}(f) =$

$\text{conv}\{(0, 0, -\text{Log}(1)), (2, 0, -\text{Log}(1)), (0, 3, -\text{Log}(1)), (1, 1, -\text{Log}(100))\}$

Archimedean Newton Polytope

Denoted by $\text{ArchNewt}(f)$

$\text{ArchNewt}(f) := \text{conv}\{(a_i, -\text{Log}|c_i|) \mid i \in \{1, \ldots, t\}, c_i \neq 0\}$

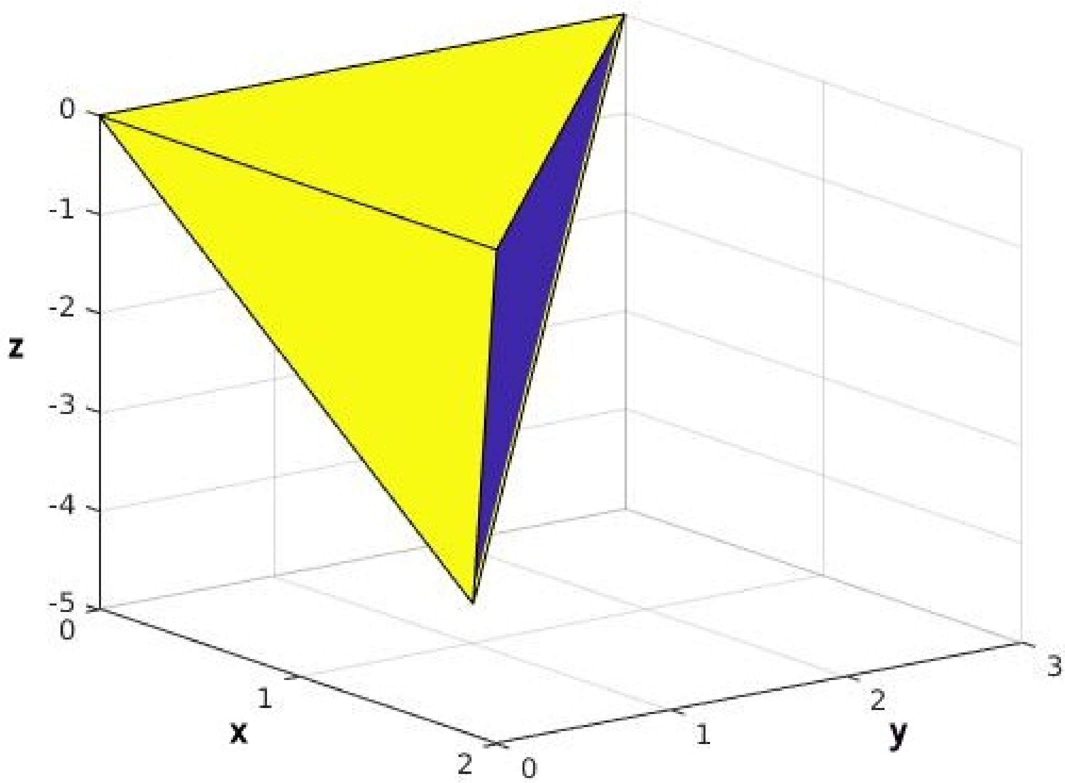
$f(x, y) = 1 * x^0 * y^0 + 1 * x^2 * y^0 + 1 * x^0 * y^3 - 100 * x^1 * y^1$

$\Rightarrow \text{ArchNewt}(f) =$

$\text{conv}\{(0, 0, -\text{Log}(1)), (2, 0, -\text{Log}(1)), (0, 3, -\text{Log}(1)), (1, 1, -\text{Log}(100))\}$

$\Rightarrow \text{conv}\{(0, 0, 0), (2, 0, 0), (0, 3, 0), (1, 1, -\text{Log}(100))\}$

$\text{conv}\{(0,0,0),(2,0,0),(0,3,0),(1,1,-\text{Log}(100))\}$



Archimedean Tropical Variety

Archimedean Tropical Variety

We need two things to construct $\mathrm{ArchTrop}(f)$

Archimedean Tropical Variety

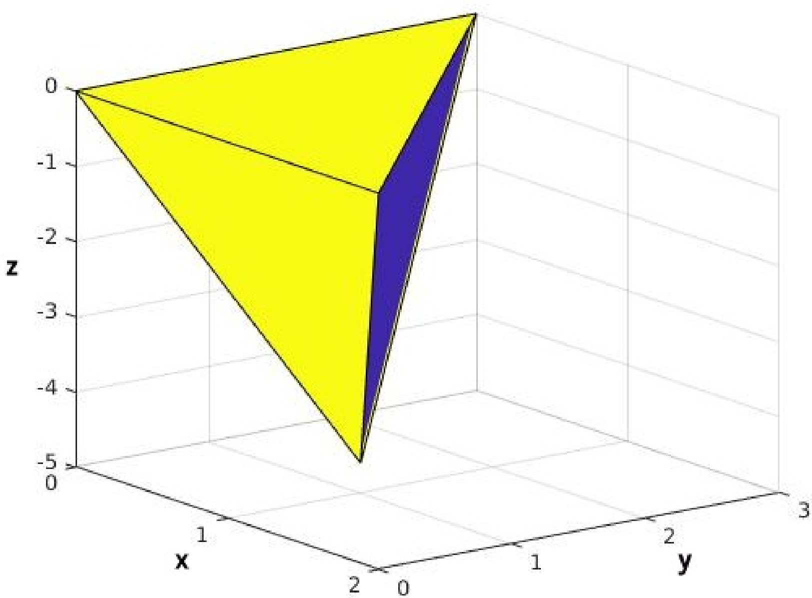
We need two things to construct ArchTrop(f)

$\rightarrow$ The outer normals of ArchNewt(f) that point downwards

Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

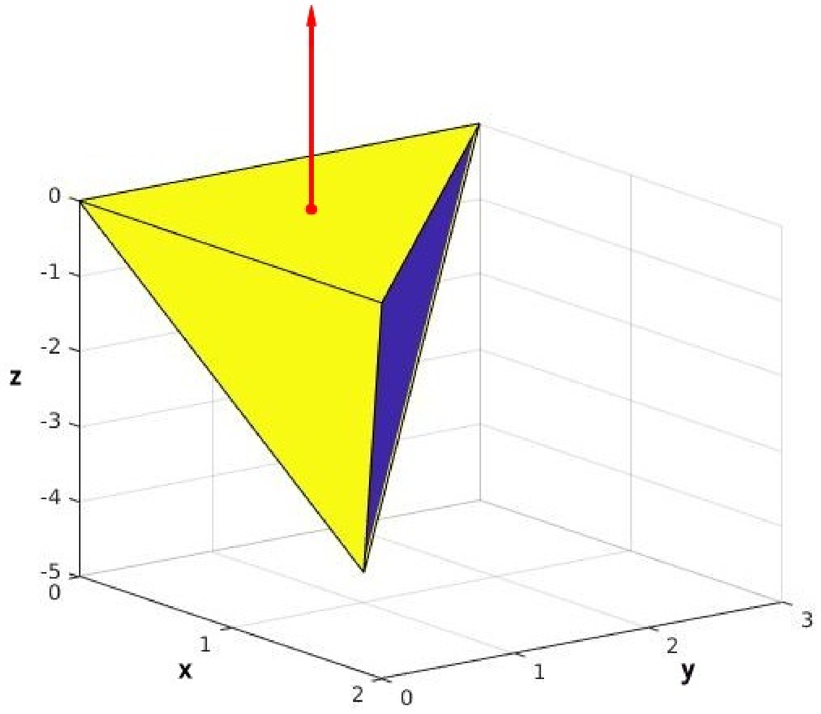
→ The outer normals of ArchNewt(f) that point downwards



Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

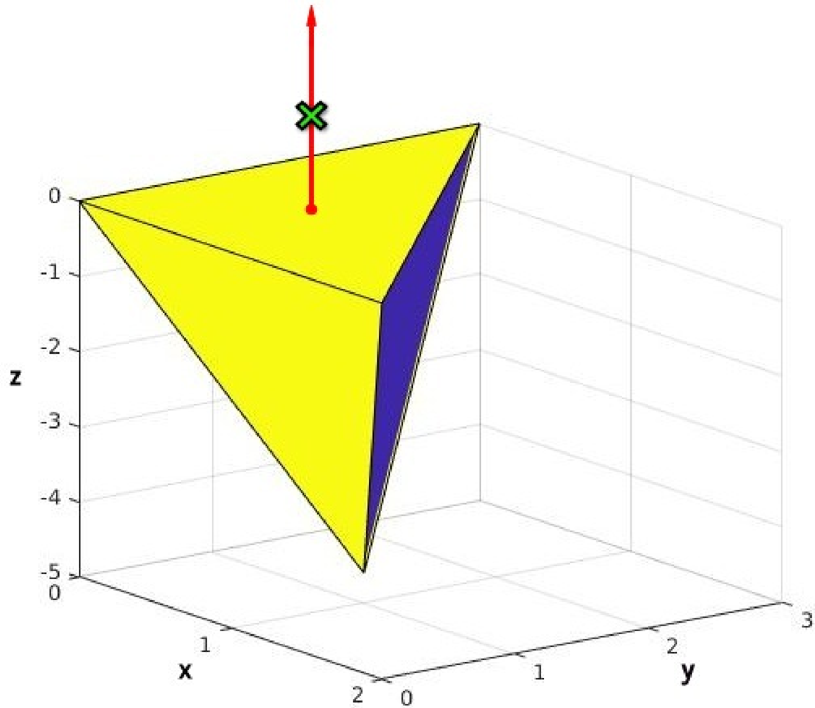
→ The outer normals of ArchNewt(f) that point downwards



Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

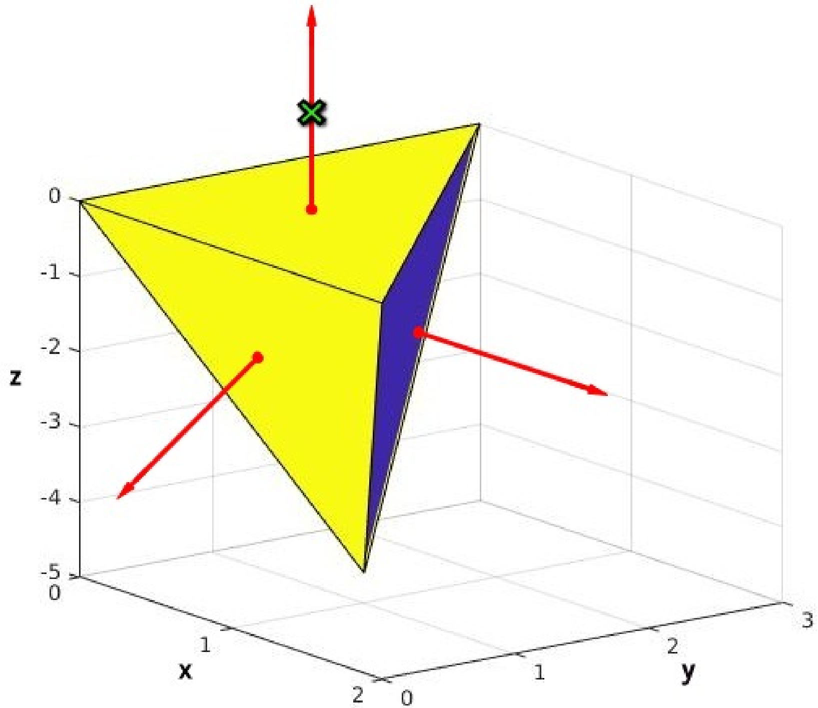
→ The outer normals of ArchNewt(f) that point downwards



Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

→ The outer normals of ArchNewt(f) that point downwards

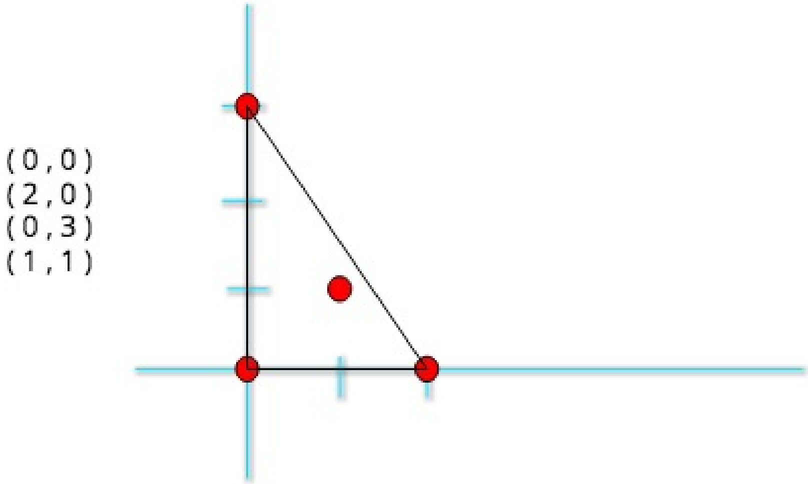


Archimedean Tropical Variety

Let's project the lower faces of ArchNewt(f) onto the xy-plane

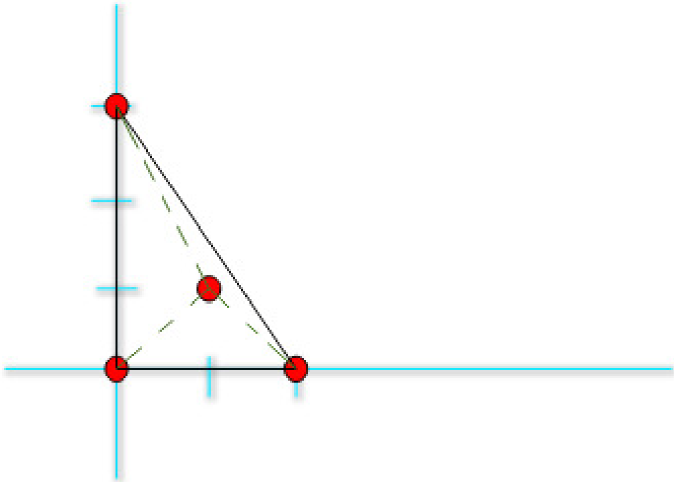
Archimedean Tropical Variety

Let's project the lower faces of ArchNewt(f) onto the xy-plane



Archimedean Tropical Variety

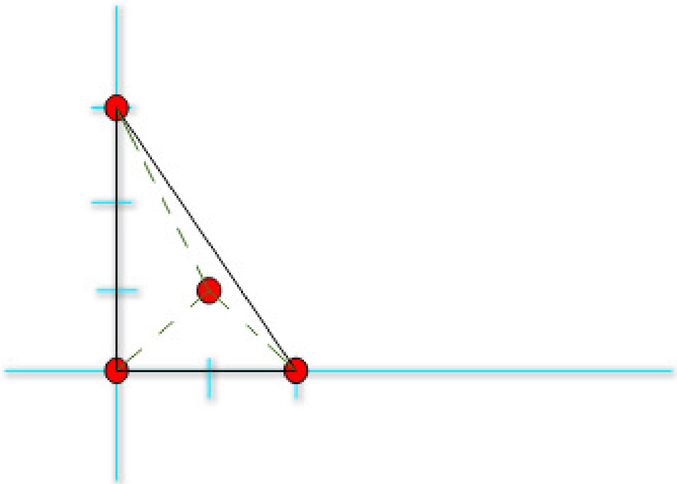
Let's project the lower faces of ArchNewt(f) onto the xy-plane



Archimedean Tropical Variety

Let's project the lower faces of ArchNewt(f) onto the xy-plane

This gives us a *triangulation* of our Newton Polytope!

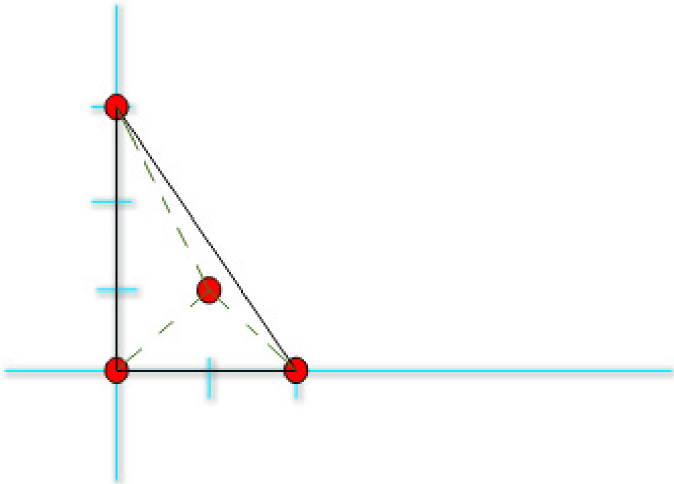


Archimedean Tropical Variety

Let's project the lower faces of ArchNewt(f) onto the xy-plane

This gives us a *triangulation* of our Newton Polytope!

We take the outer normals of these lower faces

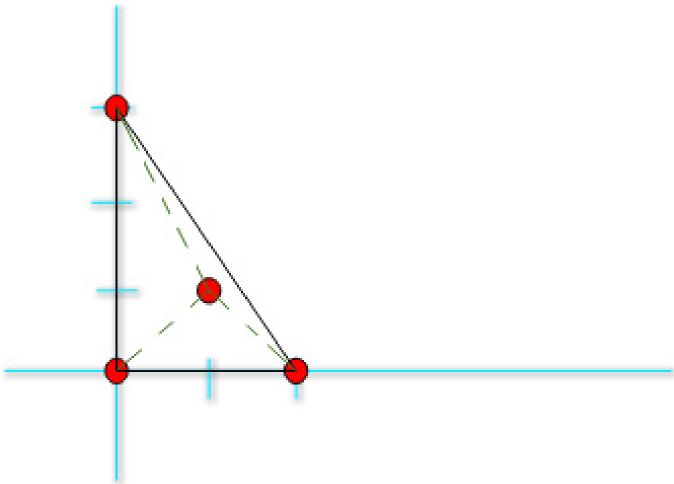


Let's project the lower faces of ArchNewt(f) onto the xy-plane

This gives us a *triangulation* of our Newton Polytope!

We take the outer normals of these lower faces

$\rightarrow$We normalize them to be of the form $(w, -1)$, and take w to be a vertex of ArchTrop(f)!



Archimedean Tropical Variety

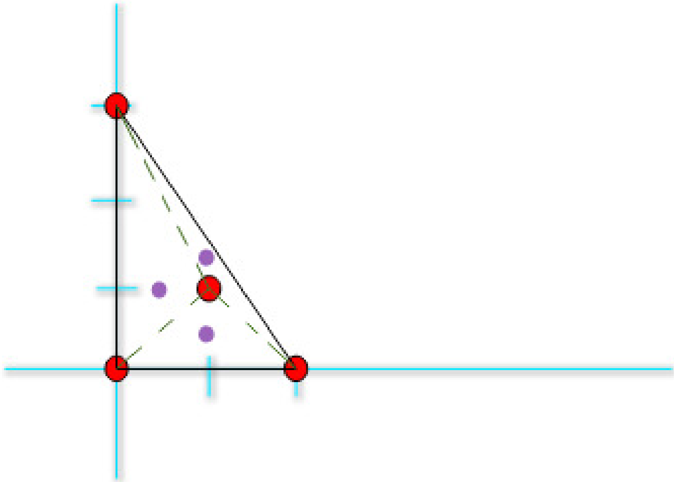
Let's project the lower faces of ArchNewt(f) onto the xy-plane

This gives us a *triangulation* of our Newton Polytope!

We take the outer normals of these lower faces

$\rightarrow$We normalize them to be of the form $(w, -1)$, and take w to be a vertex of ArchTrop(f)!

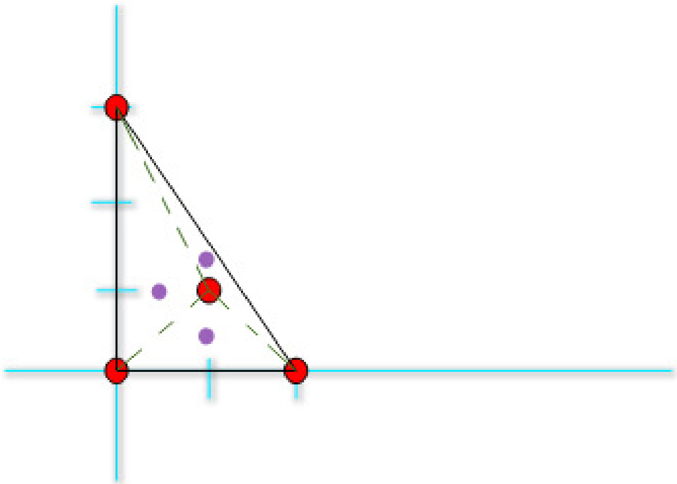
$\rightarrow$ *Roughly** translates to a point in each triangle!



Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

$\rightarrow$ The outer normals of ArchNewt(f) that point downwards

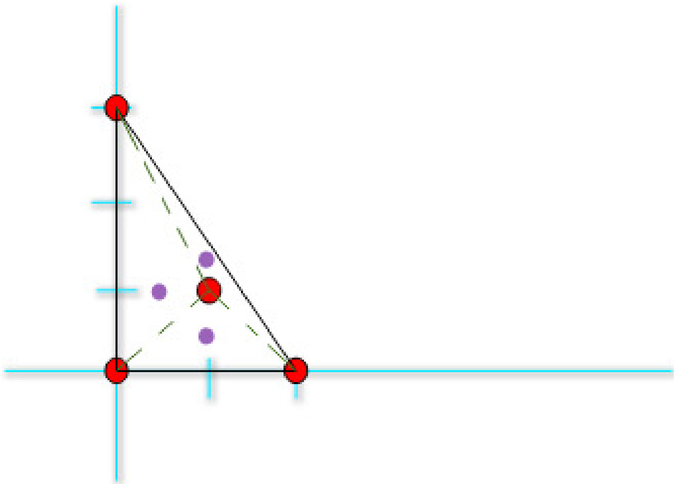


Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

→ The outer normals of ArchNewt(f) that point downwards

→ The outer normals of the edges of Newt(f)

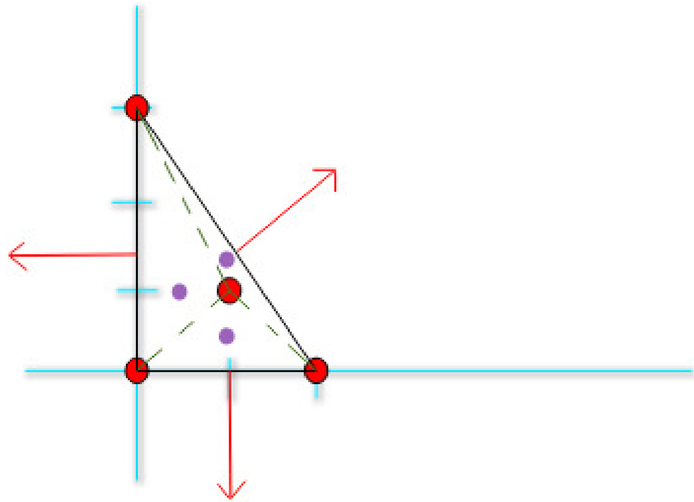


Archimedean Tropical Variety

We need two things to construct ArchTrop(f)

→ The outer normals of ArchNewt(f) that point downwards

→ The outer normals of the exterior edges of Newt(f)

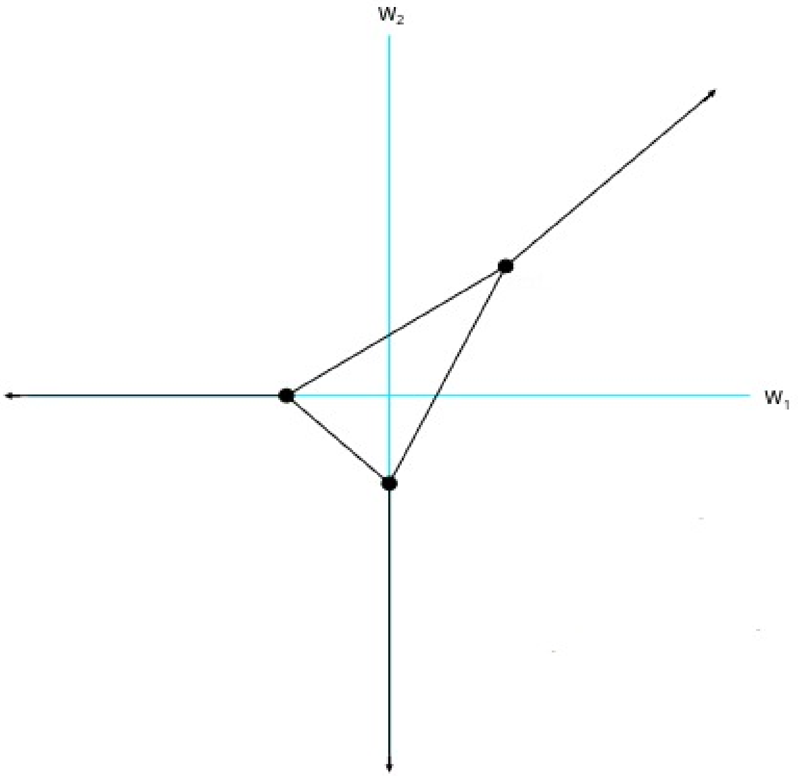


Archimedean Tropical Variety

Putting these two together, we get...

Archimedean Tropical Variety

Putting these two together, we get...



Archimedean Tropical Variety - *Roughly**

Archimedean Tropical Variety - *Roughly**

The vertices of ArchTrop(f) are *dual* to the triangulation of Newt(f) induced by the lower faces of ArchNewt(f)

Archimedean Tropical Variety - *Roughly**

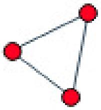
The vertices of $\text{ArchTrop}(f)$ are *dual* to the triangulation of $\text{Newt}(f)$ induced by the lower faces of $\text{ArchNewt}(f)$

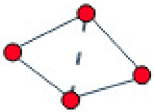
The rays are *dual* to the edges of $\text{Newt}(f)$

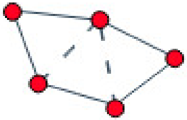
Archimedean Tropical Variety - *Roughly**

The vertices of ArchTrop(f) are *dual* to the triangulation of Newt(f) induced by the lower faces of ArchNewt(f)

The rays are *dual* to the edges of Newt(f)



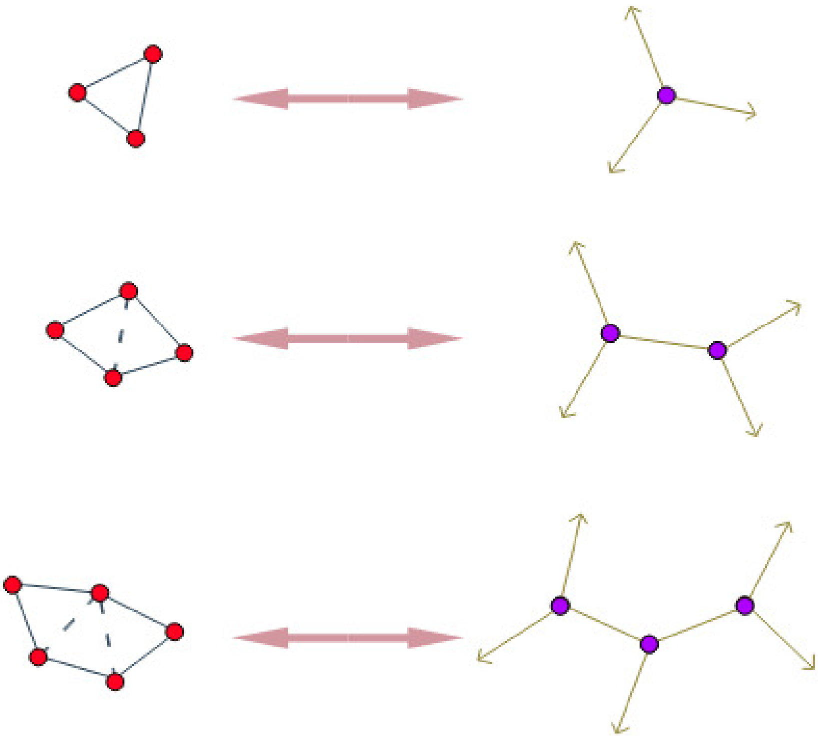




Archimedean Tropical Variety - *Roughly**

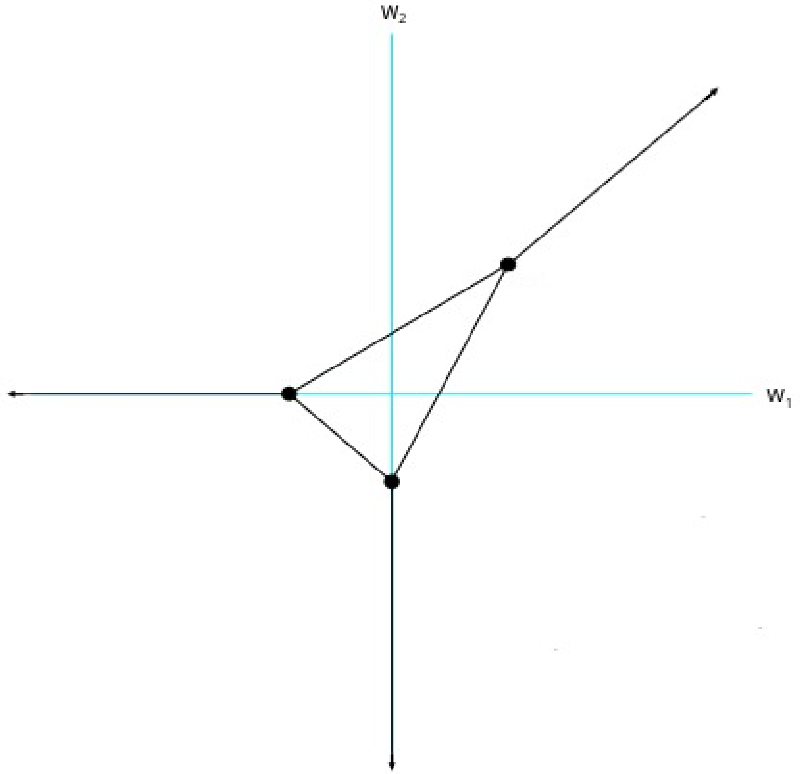
The vertices of $\mathrm{ArchTrop}(f)$ are *dual* to the triangulation of $\mathrm{Newt}(f)$ induced by the lower faces of $\mathrm{ArchNewt}(f)$

The rays are *dual* to the edges of $\mathrm{Newt}(f)$



Archimedean Tropical Variety

ArchTrop(f) gives us metric information about the roots and areas where we can find constant isotopy types!



A Word on Isotopy Types

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

Ex:
$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

Ex:

$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

$\Rightarrow$ The zero set of $f(x, y)$ is either $\varnothing$, a point, or an oval!

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

<u>**Ex**</u>:

$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

$\Rightarrow$ The zero set of $f(x, y)$ is either $\varnothing$, a point, or an oval!

$\Rightarrow$ This occurs when $c < \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, $c = \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, $c > \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, respectively

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

Ex:
$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

$\Rightarrow$ The zero set of $f(x, y)$ is either $\varnothing$, a point, or an oval!

$\Rightarrow$ This occurs when $c < \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, $c = \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, $c > \frac{6}{2^{\frac{1}{3}} 3^{\frac{1}{2}}}$, respectively

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

<u>**Ex**</u>:
$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

$\Rightarrow$ The zero set of $f(x, y)$ is either $\varnothing$, a point, or an oval!

$\Rightarrow$ This occurs when $c < \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, $c = \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, $c > \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, respectively

A Word on Isotopy Types

Much like how the quadratic discriminant $b^2 - 4ac$ gives us information about the number of roots

ArchTrop(f) can do this for more general curves

Ex:

$f(x, y) = 1 + x^2 + y^3 - cxy \ (c > 0)$

$\Rightarrow$ The zero set of $f(x, y)$ is either $\varnothing$, a point, or an oval!

$\Rightarrow$ This occurs when $c < \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, $c = \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, $c > \frac{6}{2^{\frac{1}{3}}3^{\frac{1}{2}}}$, respectively

$\mathrm{ArchTrop}_{+}(f)$

$\mathrm{ArchTrop}_+(f)$

$\mathrm{ArchTrop}_+(f) \subset \mathrm{ArchTrop}(f)$

$\mathrm{ArchTrop}_+(f)$

$\mathrm{ArchTrop}_+(f) \subset \mathrm{ArchTrop}(f)$

This time we focus on the signs of our coefficients!

$\mathrm{ArchTrop}_+(f)$

$\mathrm{ArchTrop}_+(f) \subset \mathrm{ArchTrop}(f)$

This time we focus on the signs of our coefficients!

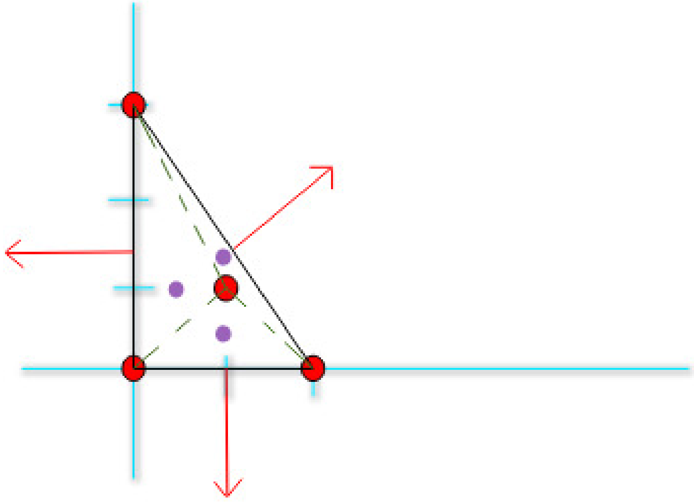
$f(x, y) = 1 + x^2 + y^3 - 100xy$

$\text{ArchTrop}_+(f)$

$\text{ArchTrop}_+(f) \subset \text{ArchTrop}(f)$

This time we focus on the signs of our coefficients!

$f(x, y) = 1 + x^2 + y^3 - 100xy$

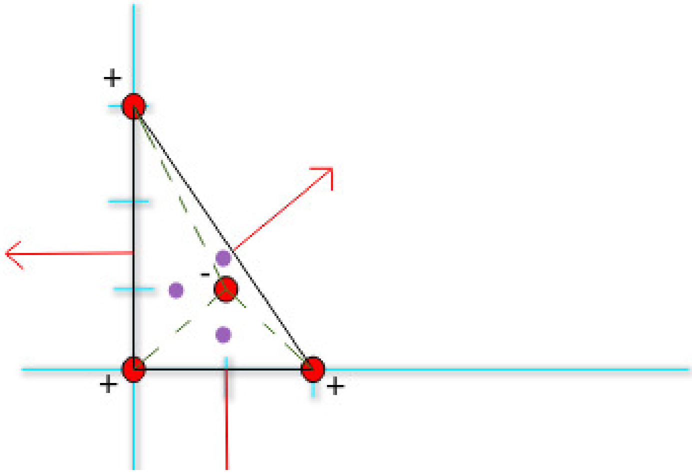


$\text{ArchTrop}_+(f)$

$\text{ArchTrop}_+(f) \subset \text{ArchTrop}(f)$

This time we focus on the signs of our coefficients!

$f(x, y) = 1 + x^2 + y^3 - 100xy$

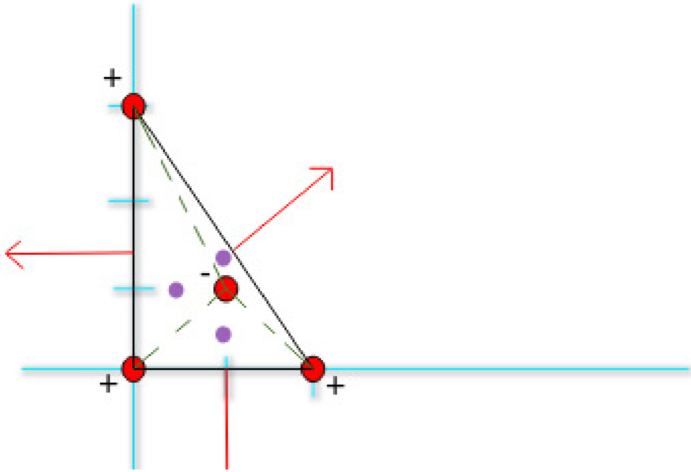


$\mathrm{ArchTrop}_+(f)$

$\mathrm{ArchTrop}_+(f) \subset \mathrm{ArchTrop}(f)$

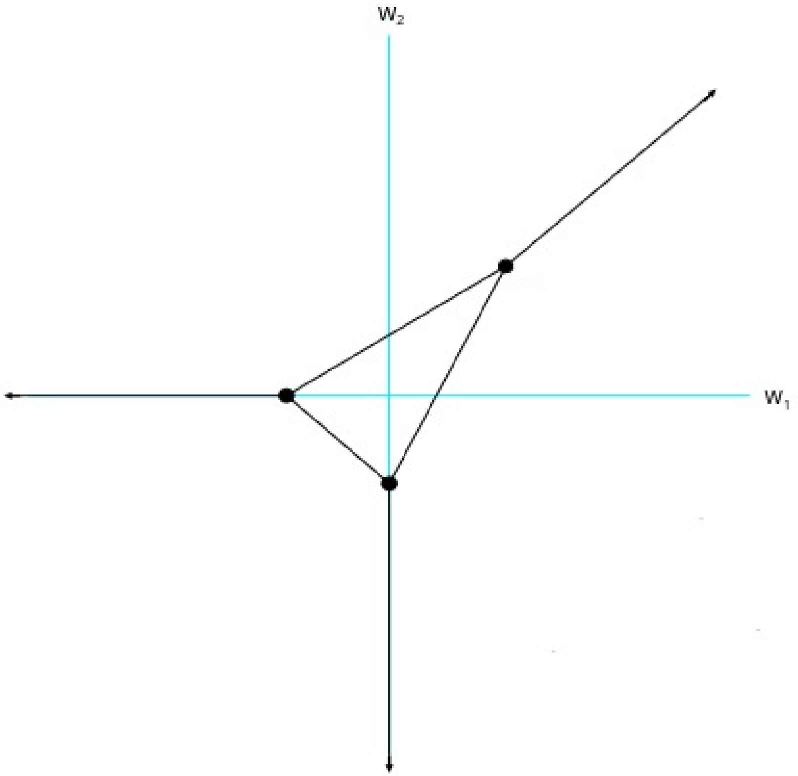
This time we focus on the signs of our coefficients!

$f(x, y) = 1 + x^2 + y^3 - 100xy$



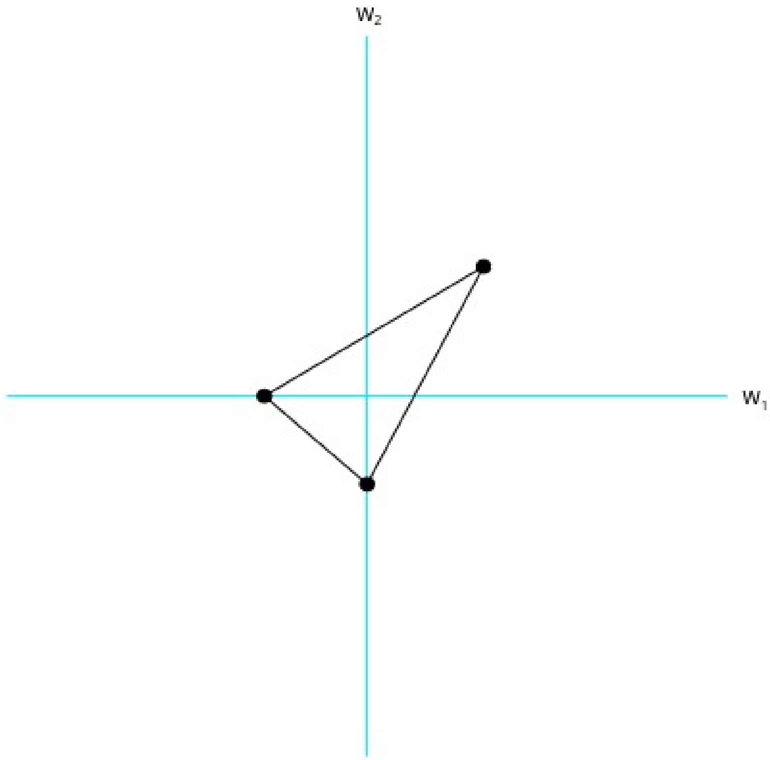
More specifically, we are interested in alternating signs!

We go from this...



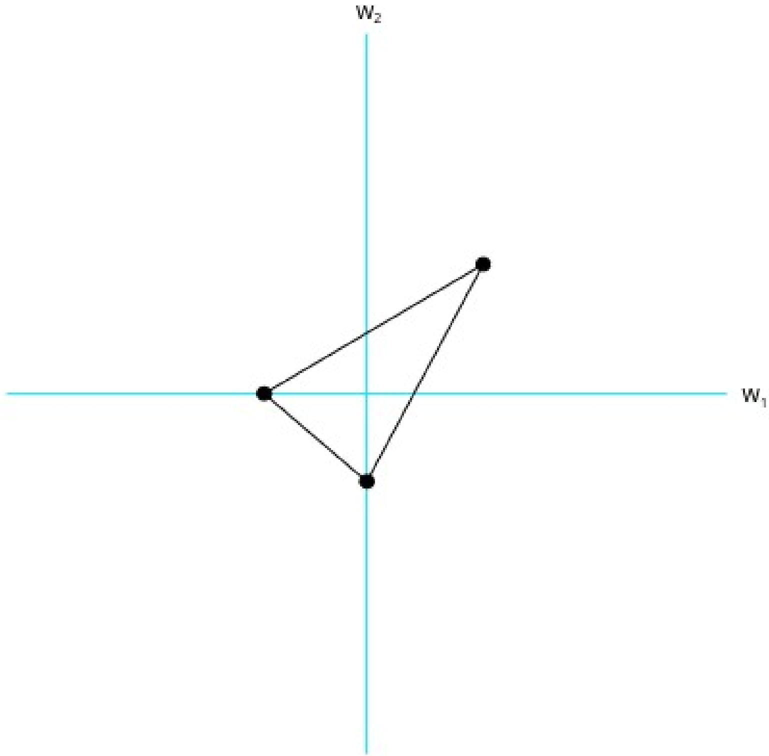
Archimedean Tropical Variety

To this!



Archimedean Tropical Variety

$\text{ArchTrop}_+(f)$ gives us a piecewise linear function that resembles the set of positive roots



A Small Discrepancy...

$$f(x) = 1 - 1.1x + x^2$$

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

$\text{ArchTrop}_+(f)$ looks like

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

$\text{ArchTrop}_+(f)$ looks like

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

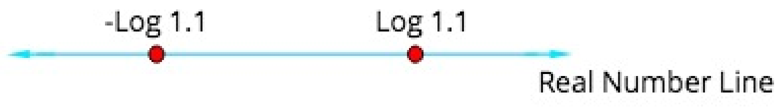
$\text{ArchTrop}_+(f)$ looks like

But if you look at the discriminant
$\Rightarrow 1.1^2 - 4 < 0 \Rightarrow f$ has two non-$\mathbb{R}$ roots!

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

$\text{ArchTrop}_+(f)$ looks like



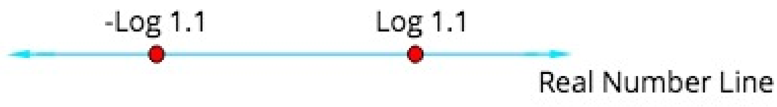
But if you look at the discriminant
$\Rightarrow 1.1^2 - 4 < 0 \Rightarrow f$ has two non-$\mathbb{R}$ roots!

On the other hand, if you look at

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

$\text{ArchTrop}_+(f)$ looks like



But if you look at the discriminant
$\Rightarrow 1.1^2 - 4 < 0 \Rightarrow f$ has two non-$\mathbb{R}$ roots!

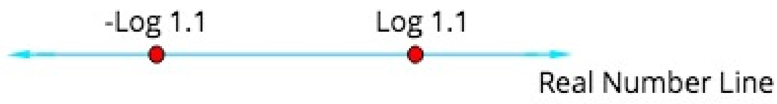
On the other hand, if you look at

$\{c \in \mathbb{R}_+ \mid \text{connected zero set of } (1 - cx + x^2) \neq \text{ArchTrop}_+(f)\}$

A Small Discrepancy...

$f(x) = 1 - 1.1x + x^2$

$\text{ArchTrop}_+(f)$ looks like



But if you look at the discriminant
$\Rightarrow 1.1^2 - 4 < 0 \Rightarrow f$ has two non-$\mathbb{R}$ roots!

On the other hand, if you look at

$\{c \in \mathbb{R}_+ \mid \text{connected zero set of } (1 - cx + x^2) \neq \text{ArchTrop}_+(f)\}$

$= (0, 2)$

Our Research

Our Research - Newton Polytope

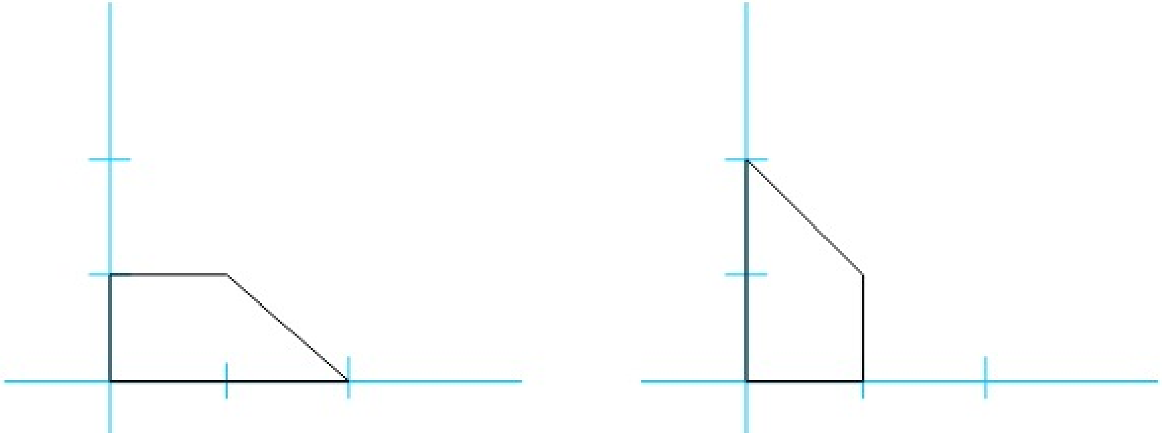
$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

$$f_2(x_8, x_9) = c_6 x_9^2 + c_7 x_8 x_9 + c_8 x_8 + c_9 x_9 + c_{10}$$

Our Research - Newton Polytope

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

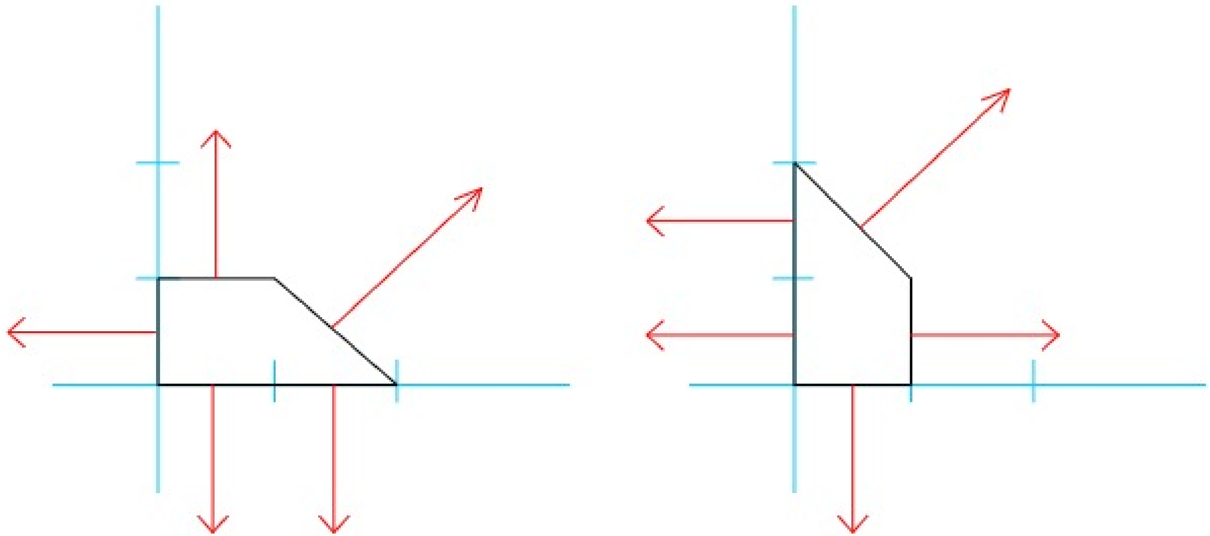
$$f_2(x_8, x_9) = c_6 x_9^2 + c_7 x_8 x_9 + c_8 x_8 + c_9 x_9 + c_{10}$$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

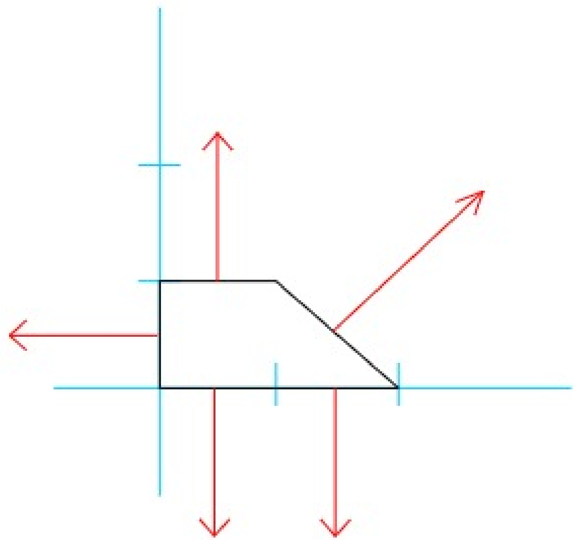
$$f_2(x_8, x_9) = c_6 x_9^2 + c_7 x_8 x_9 + c_8 x_8 + c_9 x_9 + c_{10}$$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

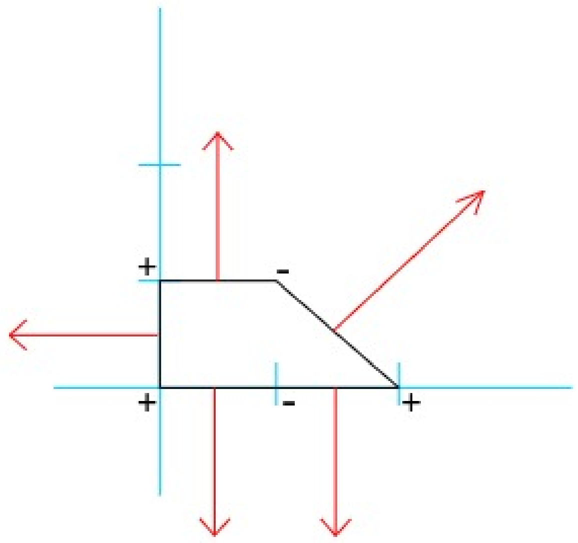
Suppose $c_1 = 1, c_2 = -10, c_3 = -10, c_4 = 2, c_5 = 1$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

Suppose $c_1 = 1, c_2 = -10, c_3 = -10, c_4 = 2, c_5 = 1$

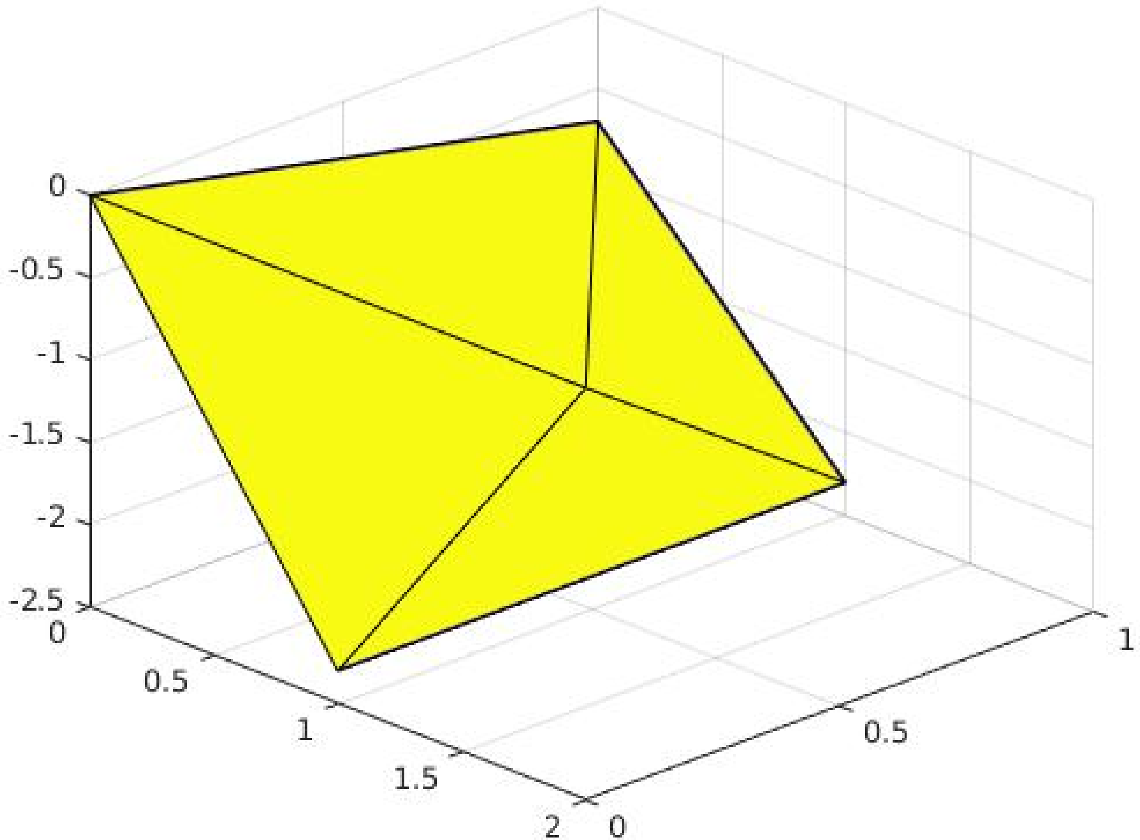


Our Research - f_1

ArchNewt(f_1) given *these* coefficients is...

Our Research - f_1

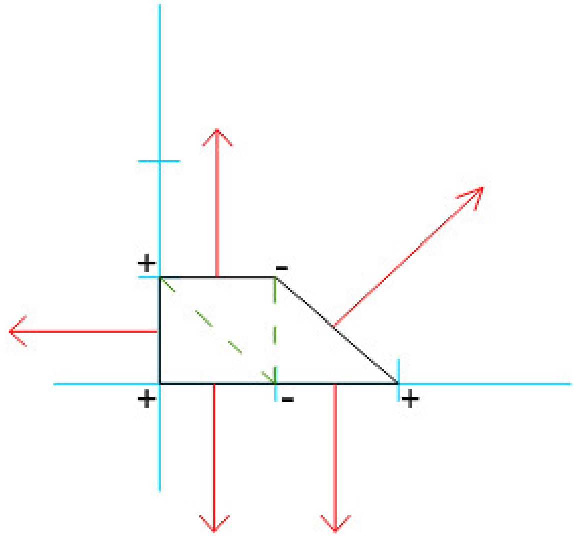
ArchNewt(f_1) given *these* coefficients is...



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

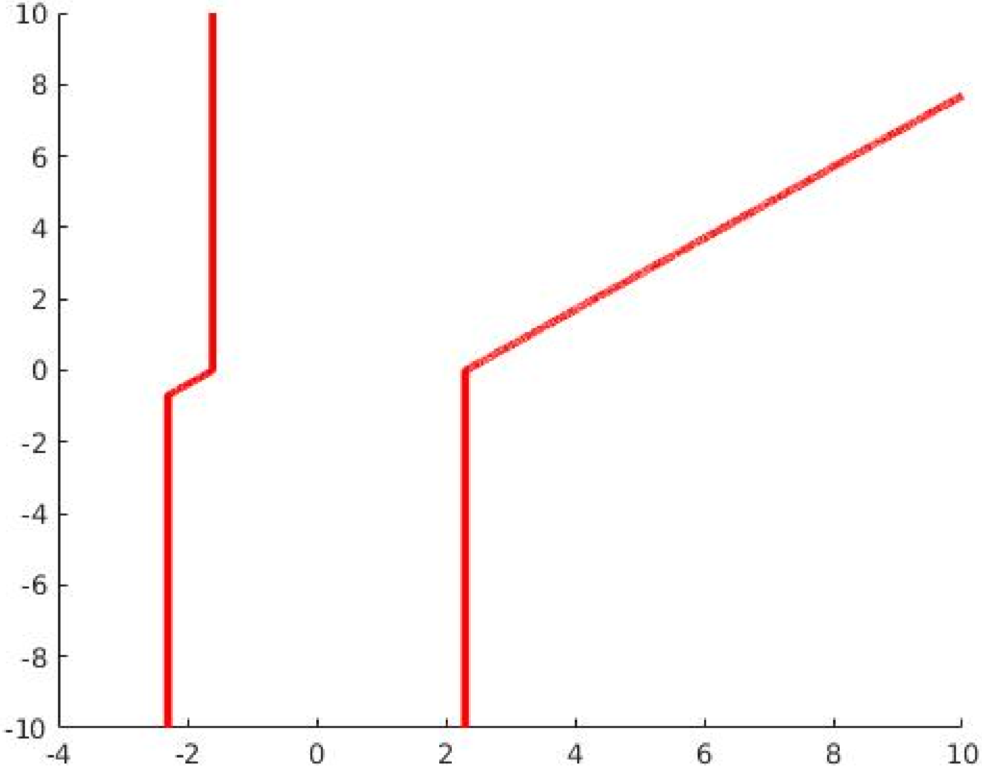
Suppose $c_1 = 1, c_2 = -10, c_3 = -10, c_4 = 2, c_5 = 1$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

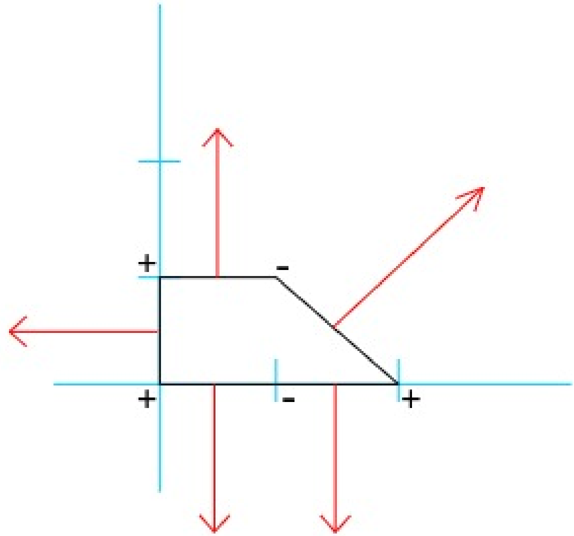
Suppose $c_1 = 1, c_2 = -10, c_3 = -10, c_4 = 2, c_5 = 1$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

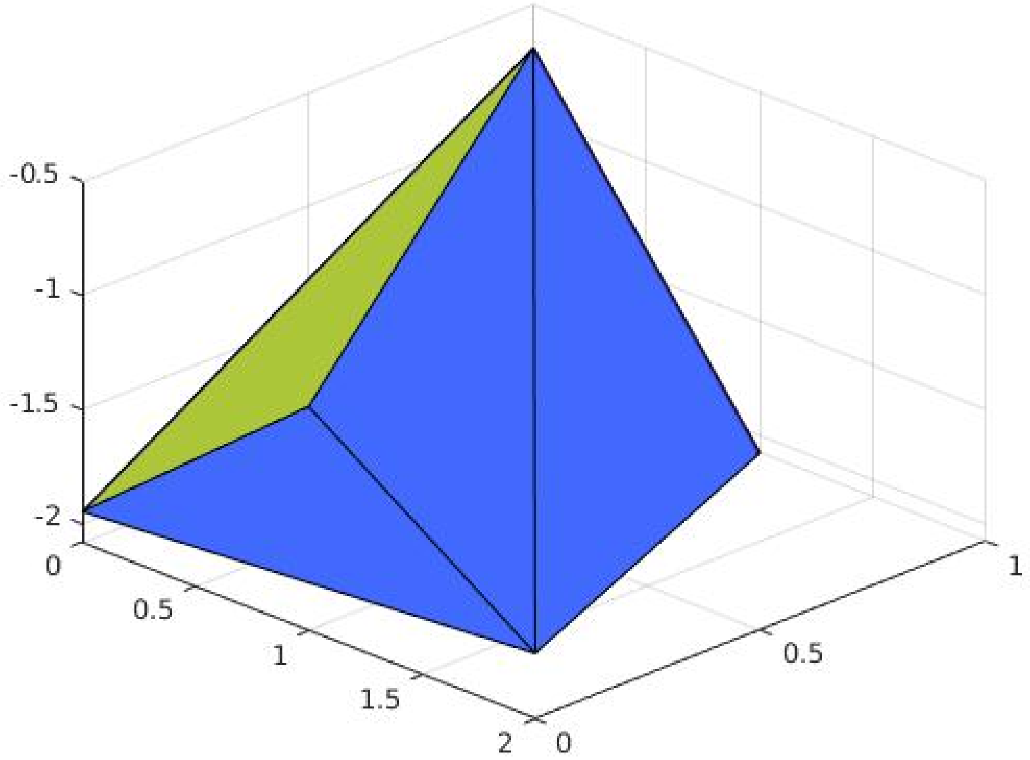
Suppose $c_1 = 6, c_2 = -8, c_3 = -3, c_4 = 2, c_5 = 7$



Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

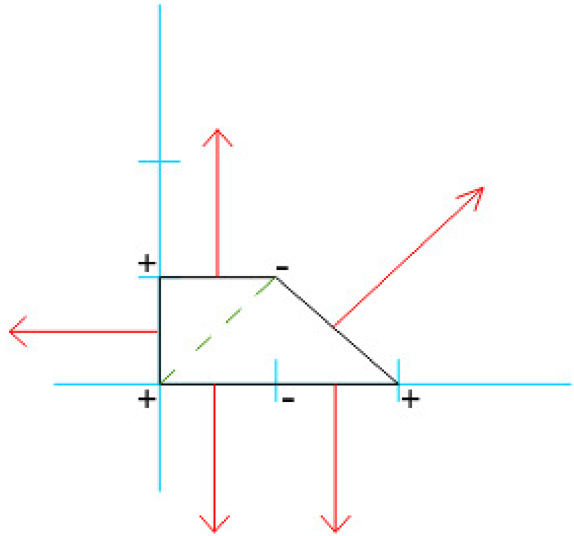
Suppose $c_1 = 6, c_2 = -8, c_3 = -3, c_4 = 2, c_5 = 7$



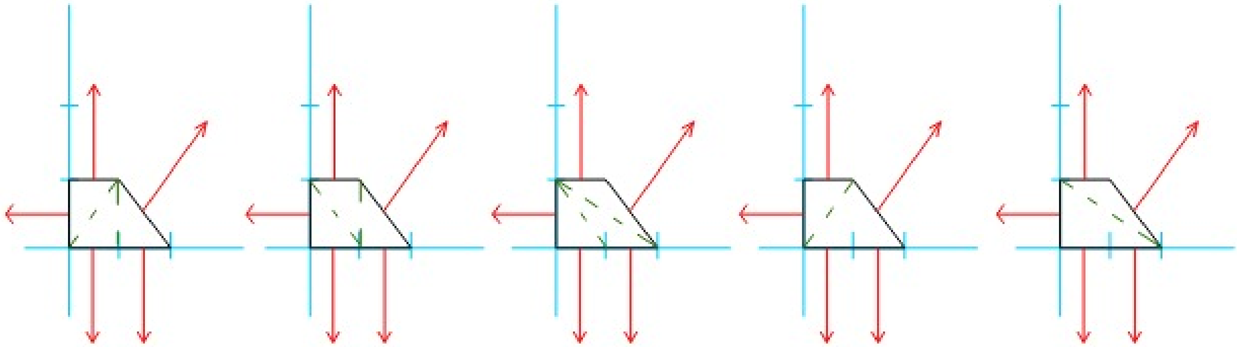
Our Research - f_1

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

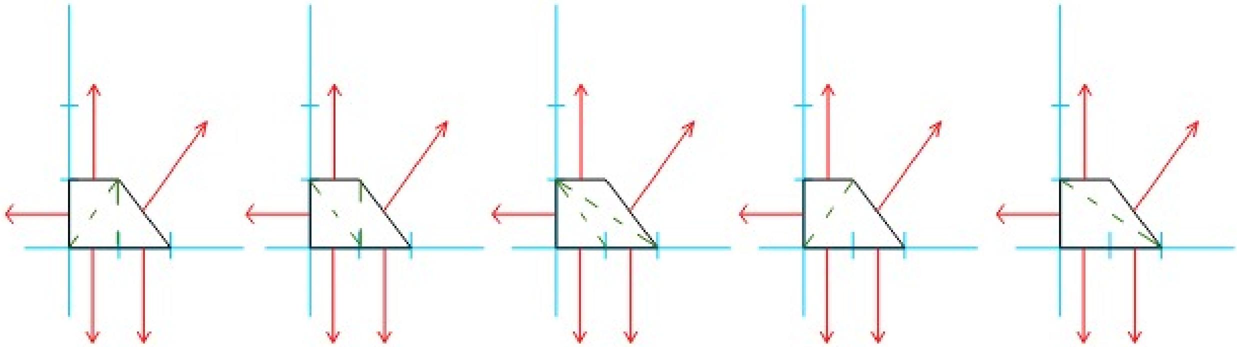
Suppose $c_1 = 6, c_2 = -8, c_3 = -3, c_4 = 2, c_5 = 7$



Our Research



Our Research

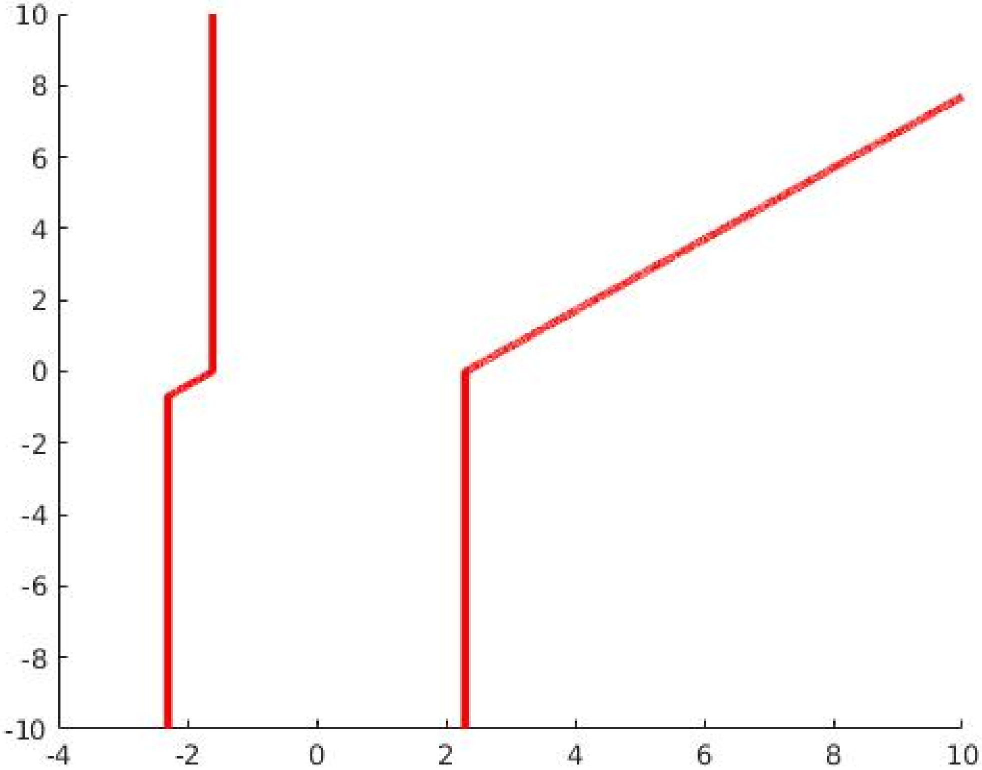


No matter the coefficients, these 5 cases encompass all the possible triangulations of f_1!

Our Research

$$f_1(x_8, x_9) = c_1 x_8^2 + c_2 x_8 x_9 + c_3 x_8 + c_4 x_9 + c_5$$

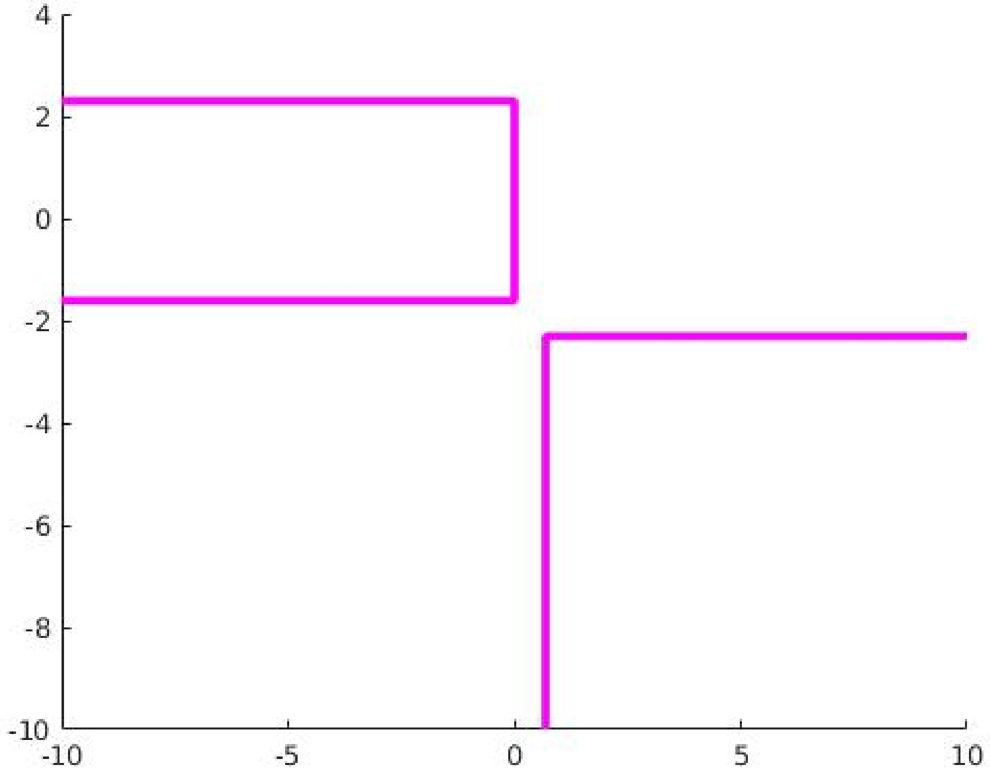
Suppose $c_1 = 1, c_2 = -10, c_3 = -10, c_4 = 2, c_5 = 1$



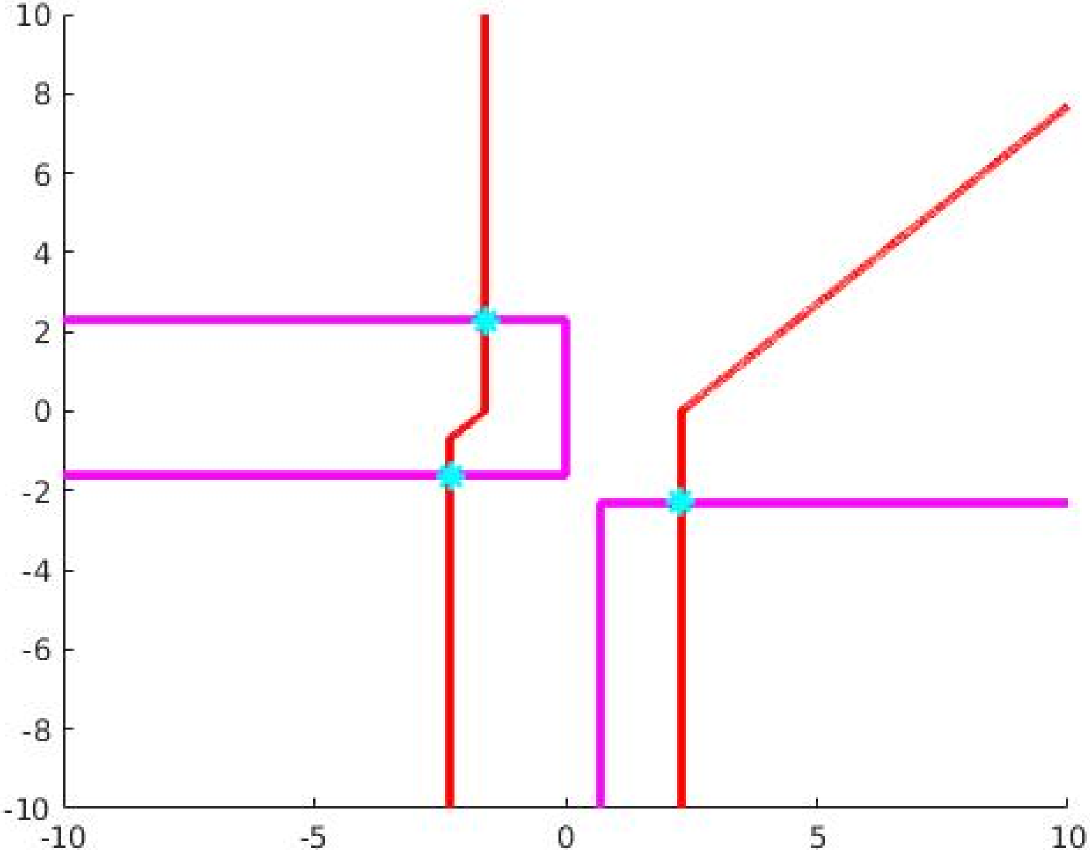
Our Research

$$f_2(x_8, x_9) = c_6 x_8 x_9 + c_7 x_8^2 + c_8 x_8 + c_9 x_9 + c_{10}$$

Suppose $c_6 = 1, c_7 = -10, c_8 = -10, c_9 = 2, c_10 = 1$



Our Research



A Theorem on ArchTrop(f)

A Theorem on ArchTrop(f)

$Z_{\mathbb{C}}(f) :=$ the Complex zero set of f

A Theorem on ArchTrop(f)

$Z_{\mathbb{C}}(f) :=$ the Complex zero set of f

Theorem

For any pentanomial f in $\mathbb{C}[x_1, \ldots, x_n]$, any point of $\mathrm{Log}|Z_{\mathbb{C}}(f)|$ is within distance $\log(4)$ of some point of ArchTrop(f).

A Theorem on ArchTrop$_+(f)$

$Z_+(f) :=$ the positive zero set of f

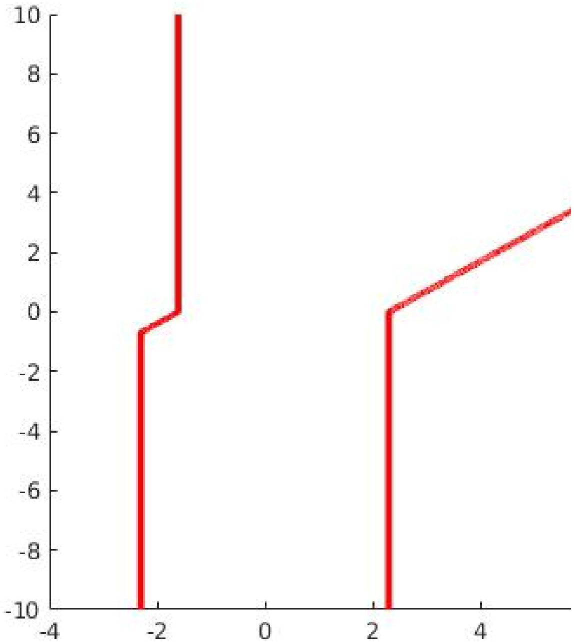
A Theorem on $\text{ArchTrop}_+(f)$

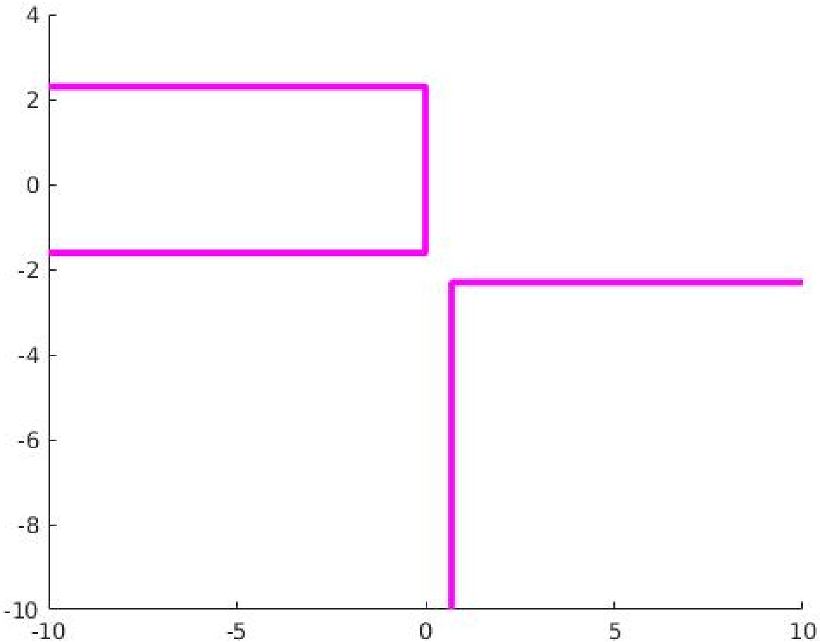
$Z_+(f) :=$ the positive zero set of f

Theorem

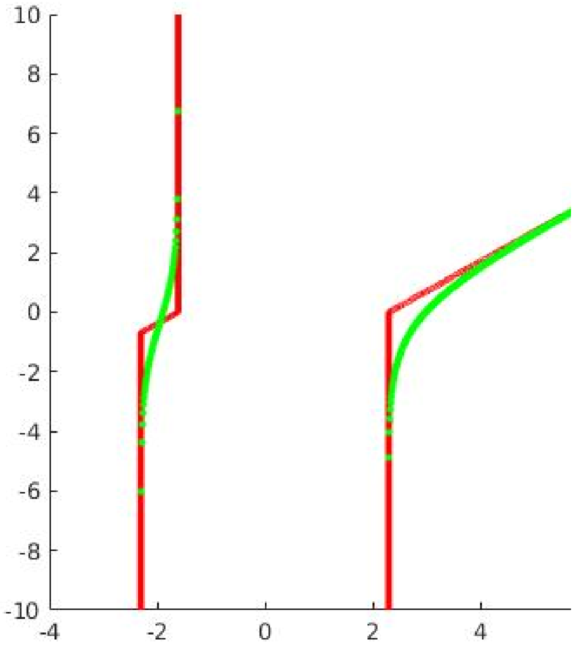
For any pentanomial f in $\mathbb{R}[x_1, \ldots, x_n]$, any point of $\text{Log}|Z_+(f)|$ is within distance $\log(4)$ of some point of $\text{ArchTrop}_+(f)$.

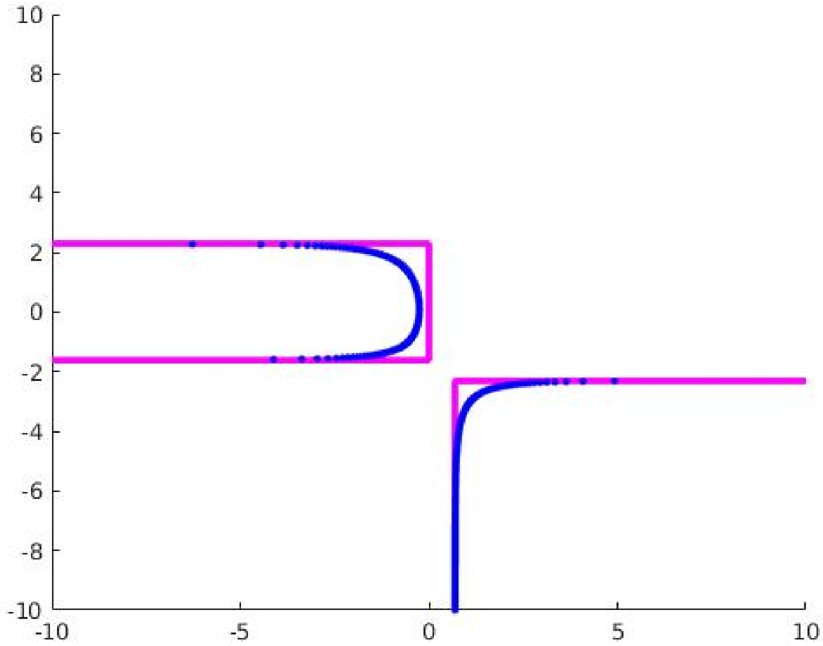
Using the same coefficients...





Using the same coefficients...



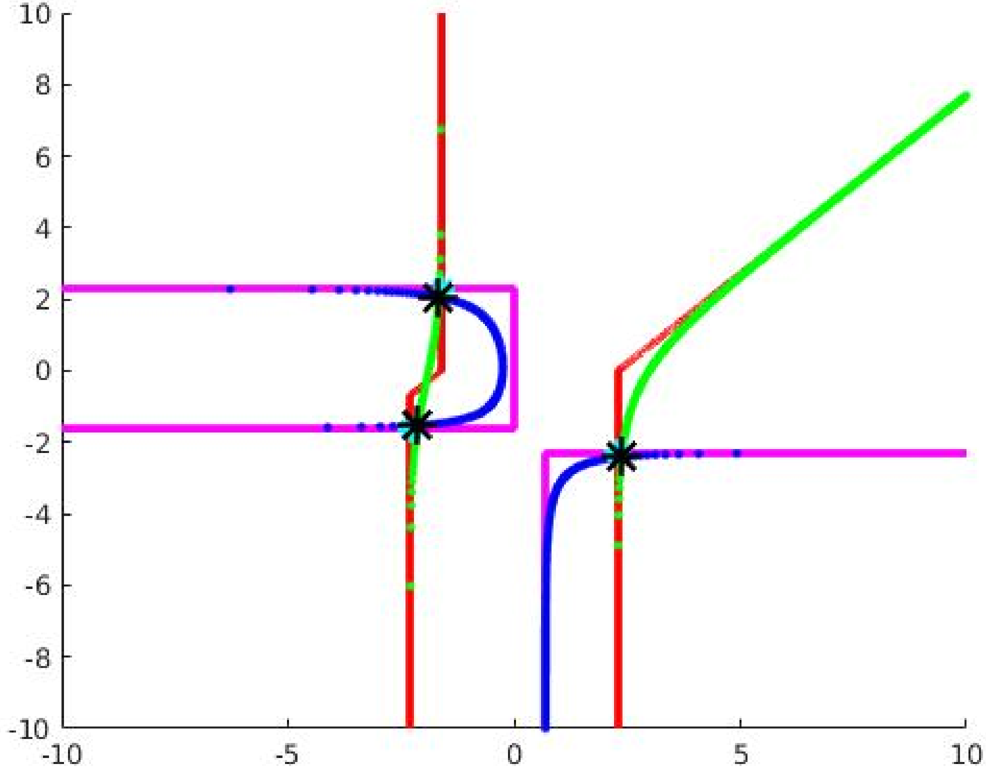


Theorem

If F is a random real 2×2 quadratic pentanomial system with supports having Cayley embedding

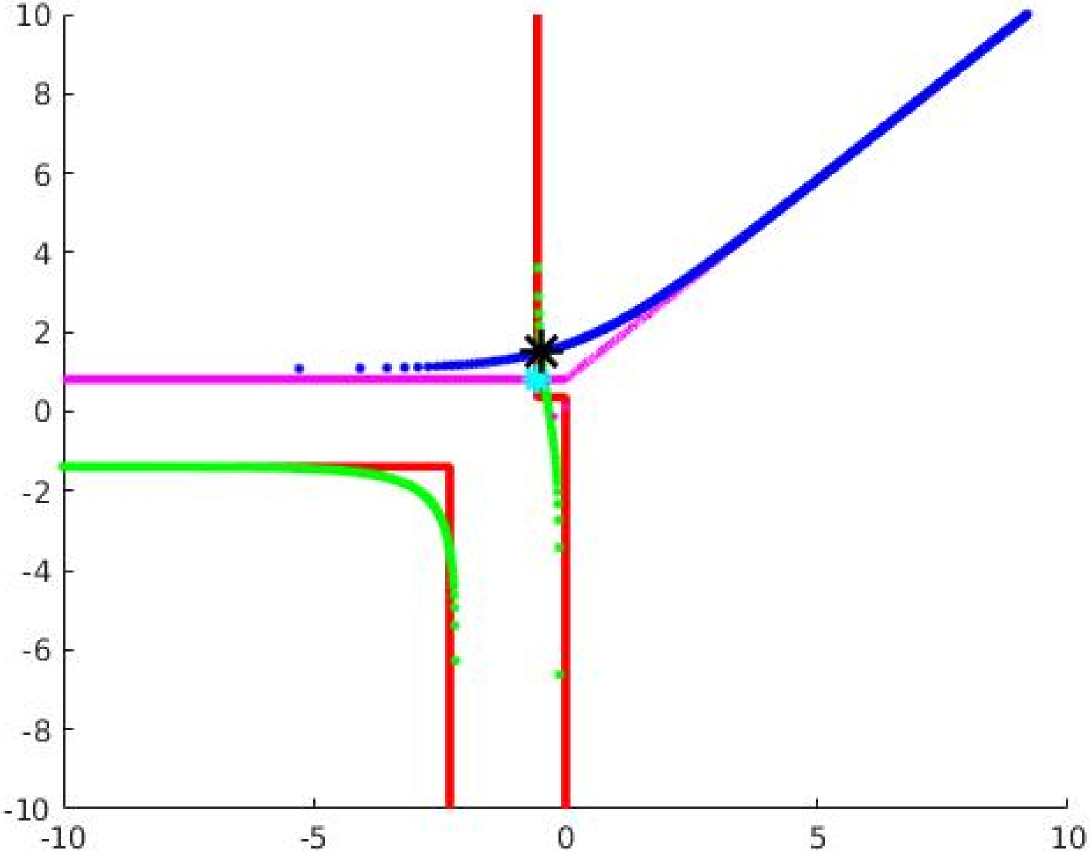
$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \end{bmatrix},$$

such that the coefficient vector $(c_1, \ldots, c_{10})$ has each c_i with mean 0, then with probability at least 41%, F has the same number of positive roots as the cardinality of $\text{ArchTrop}(f_1) \cap \text{ArchTrop}(f_2)$.

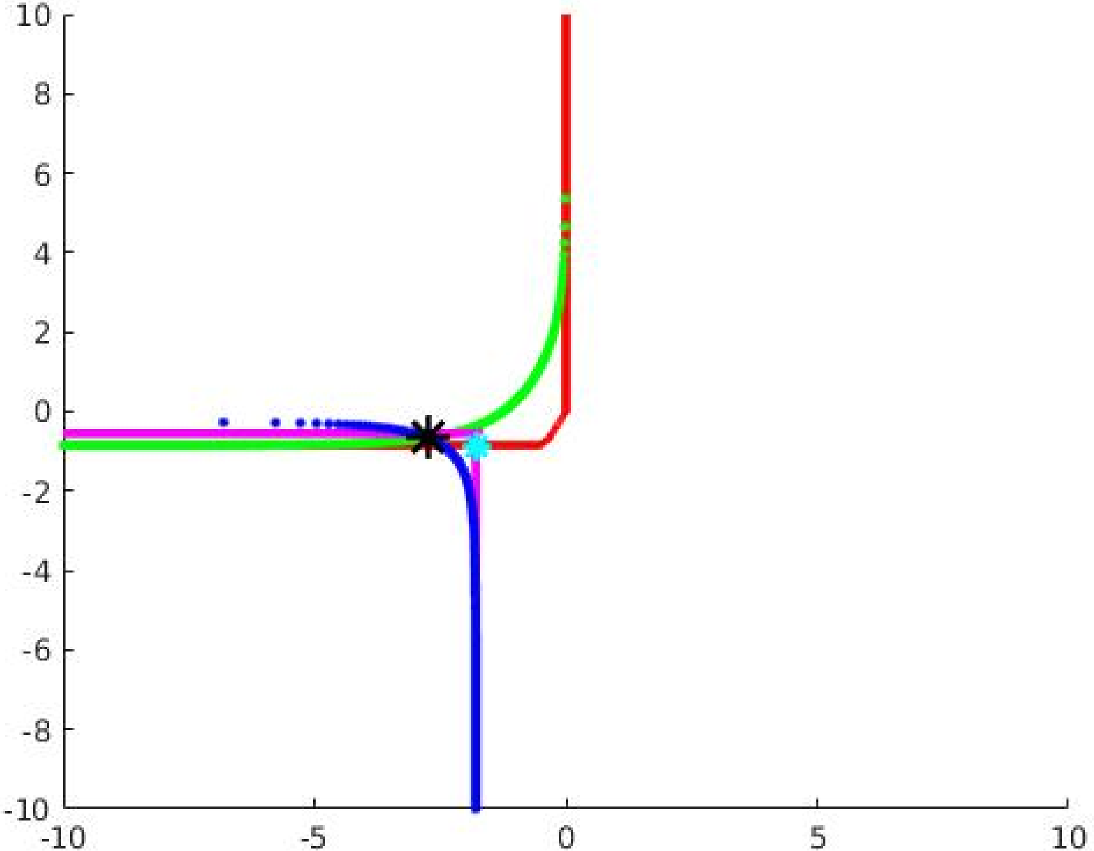


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-5
0
5
10

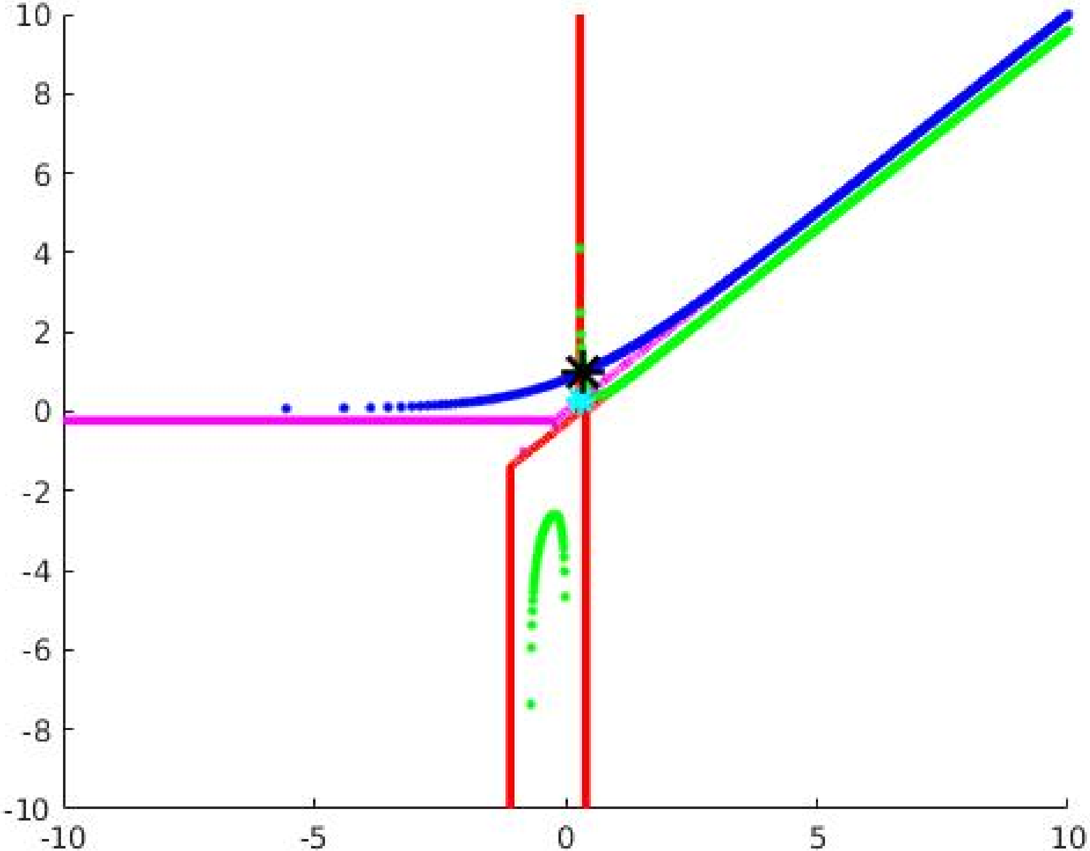
Successes!



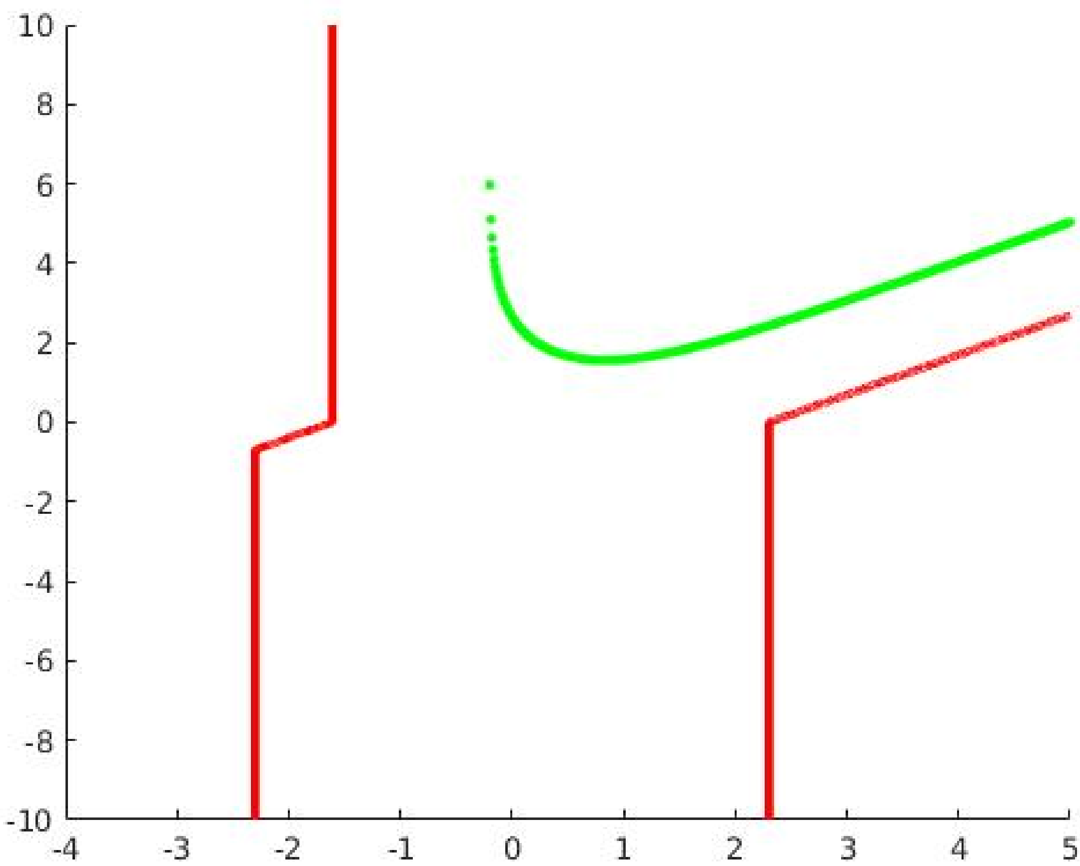
Successes!



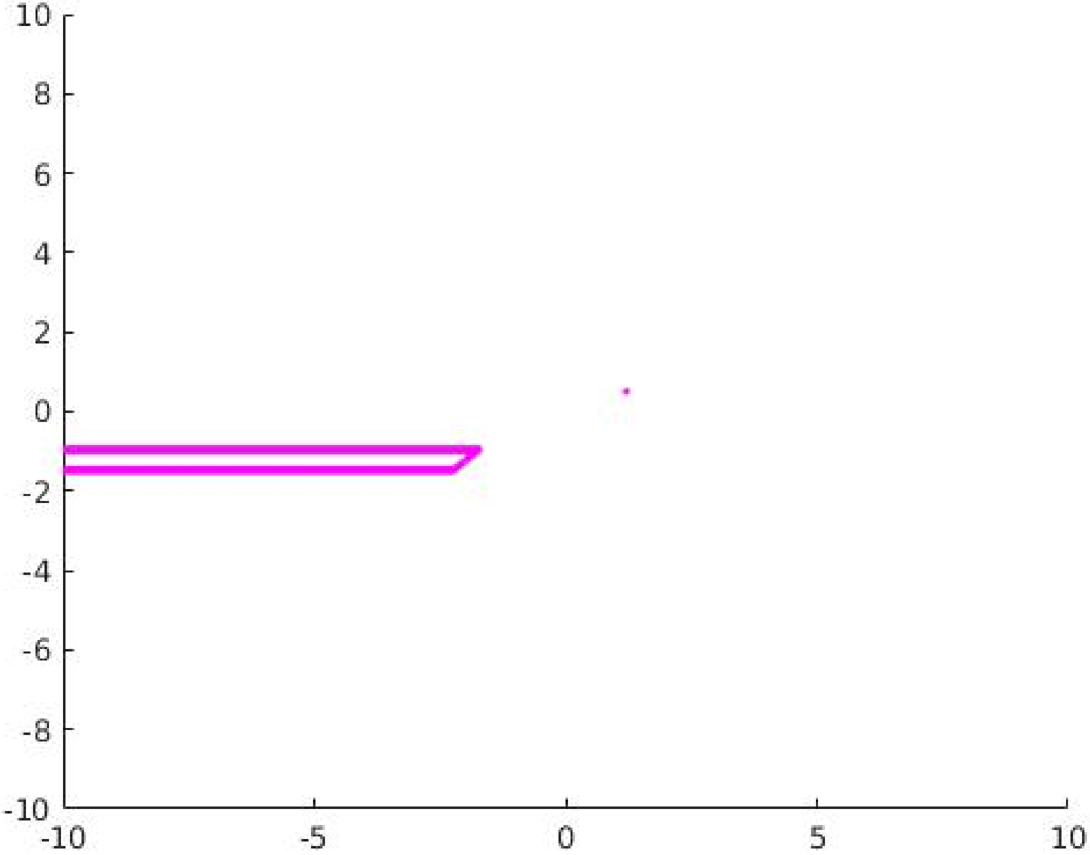
Successes!



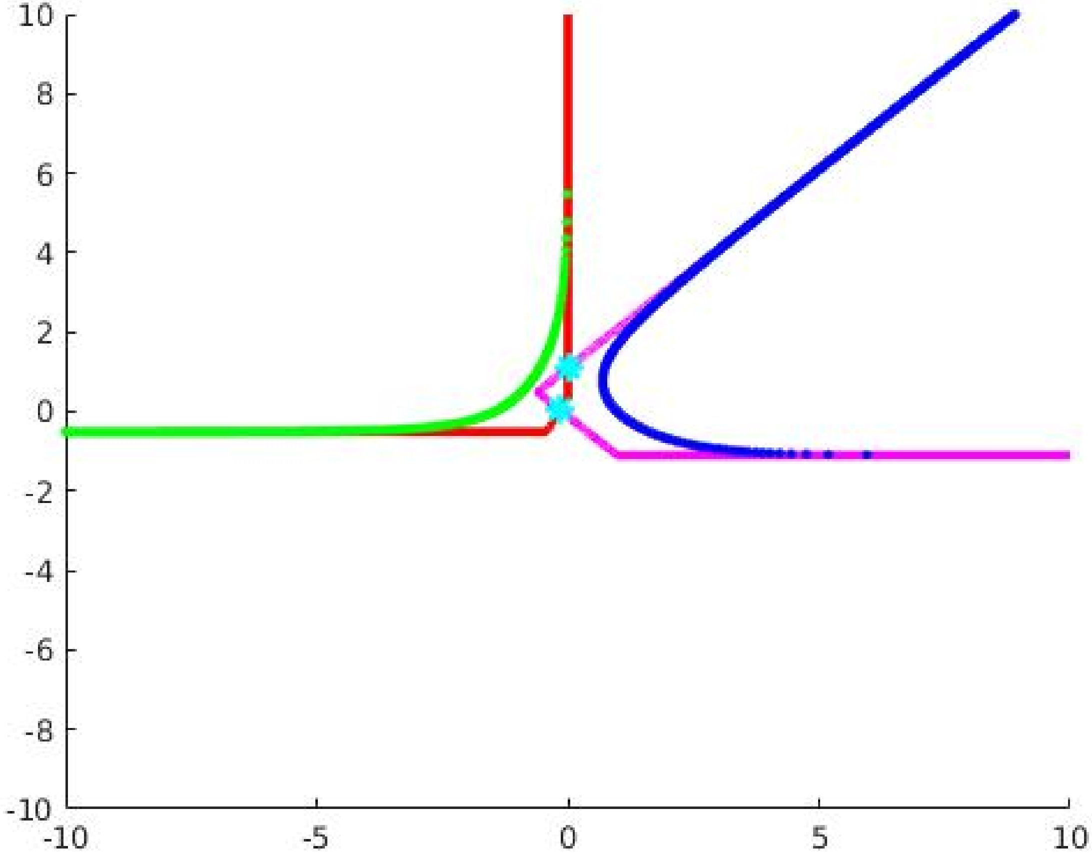
Failures...



Failures...crickets...



Failures...

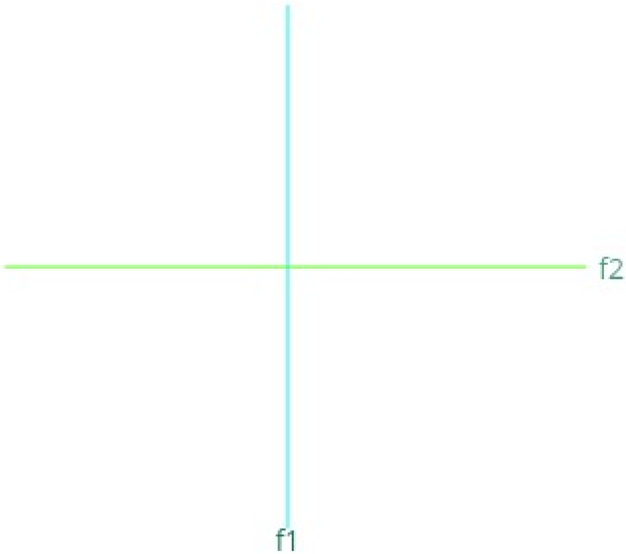


But why?

Some intuition...

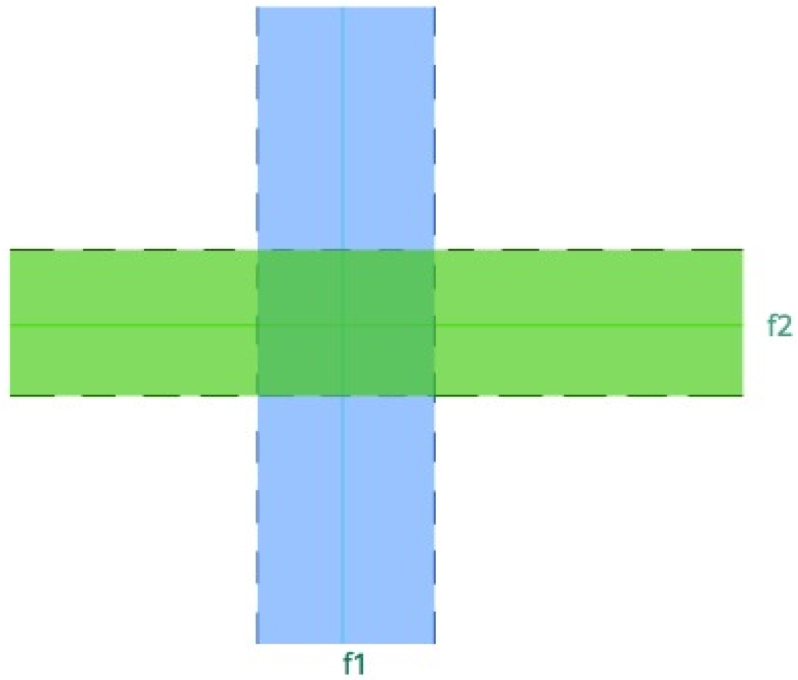
But why?

Some intuition...



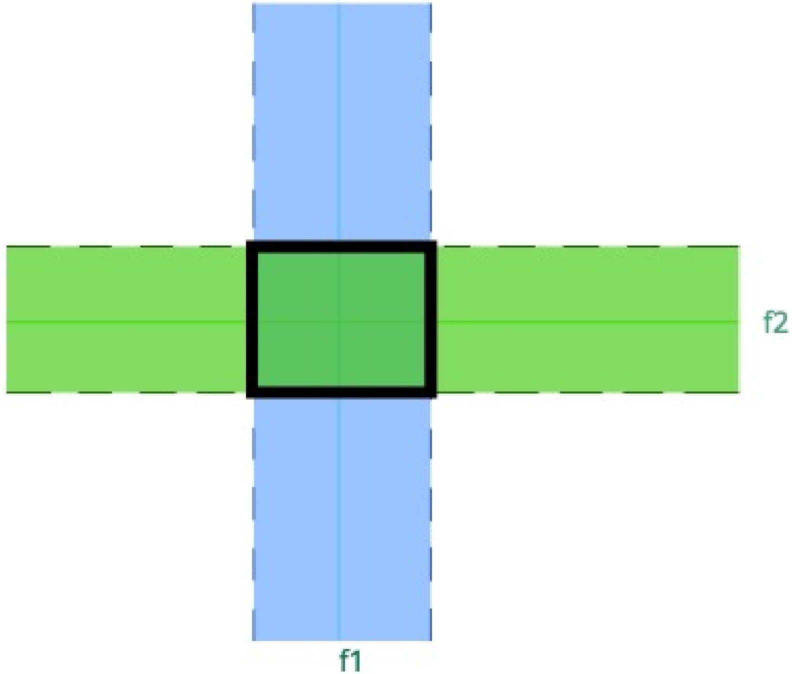
But why?

Some intuition...



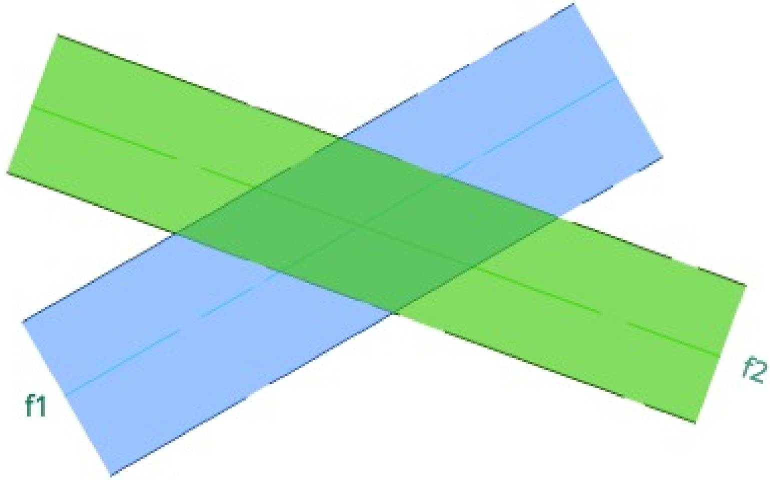
But why?

Some intuition...



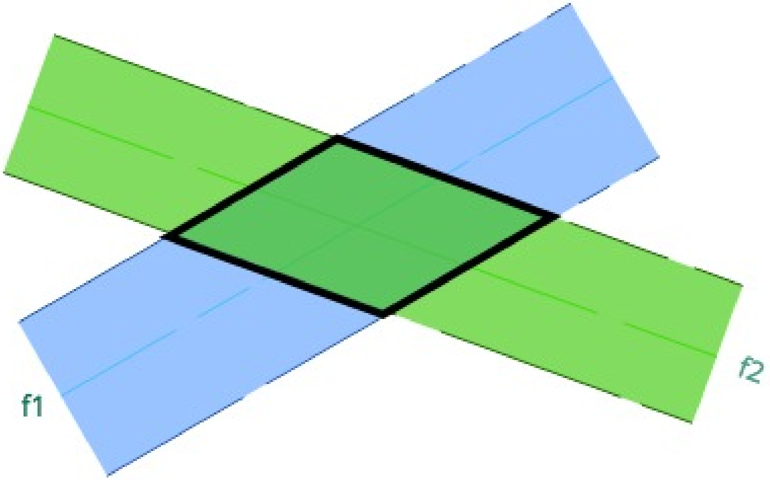
But why?

Some intuition...



But why?

Some intuition...



A Theorem on the Intersections

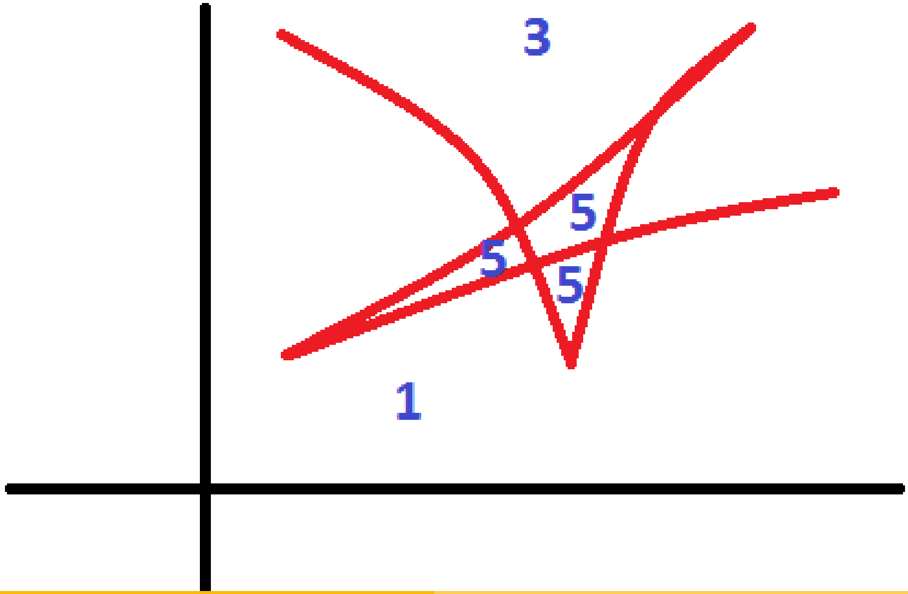
Theorem

For any 2×2 polynomial system non-degenerate F with supports having Cayley embedding A, the number of nonzero real roots of F depends only on the completed signed A-discriminant chamber containing F.

A Theorem on the Intersections

Theorem

For any 2×2 polynomial system non-degenerate F with supports having Cayley embedding A, the number of nonzero real roots of F depends only on the completed signed A-discriminant chamber containing F.



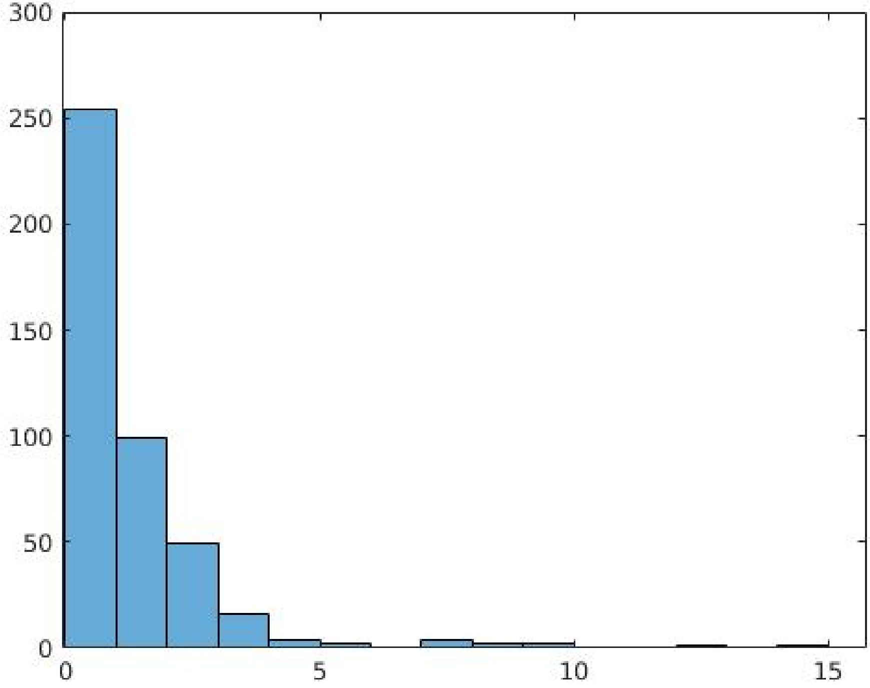
That being said...

That being said...

We can compute the Hausdorff distance between $\text{ArchTrop}(f_1) \cap \text{ArchTrop}(f_2)$ and $\text{Log}|Z_+(f_1)| \cap \text{Log}|Z_+(f_2)|$ for 1000 random examples to obtain the following:

That being said...

We can compute the Hausdorff distance between $\mathrm{ArchTrop}(f_1) \cap \mathrm{ArchTrop}(f_2)$ and $\mathrm{Log}|Z_+(f_1)| \cap \mathrm{Log}|Z_+(f_2)|$ for 1000 random examples to obtain the following:



Future Research

Future Research

1. Generalizing our code

Future Research

1. Generalizing our code
2. Finding the conditions under which

$$h_0(Z_+(f)) = h_0(\mathrm{ArchTrop}_+(f))$$

Future Research

1. Generalizing our code
2. Finding the conditions under which

$$h_0(Z_+(f)) = h_0(\mathrm{ArchTrop}_+(f))$$

3. Stability and the Jacobian