The Probability of Easily Approximating the Positive Real Roots of Trinomials

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1 Abstract

Polynomial system solving lends itself to a variety of fields, including chemical reaction networks, geolocation, and semi-definite programming. However, calculating the positive real roots for generic trinomials is inefficient. More easily calculated polytopes such as the positive tropical variety are powerful tools for approximating these roots. While the positive tropical variety is known to frequently be isotopic to the positive zero set of a trinomial, under certain conditions it is not. Knowing the probability with which these conditions are met for trinomials gives us insight into the reliability of the positive tropical variety as an approximation tool for the positive zero set. In this paper, we discuss the probability that the positive tropical variety is isotopic to the positive zero set for trinomials with normally distributed coefficients.

2 Introduction

Solving polynomial systems is the backbone for countless fields of study. In particular, they are useful for modeling non-linear systems, and therefore have application in such diverse fields as biology, and engineering and mechanics. Due to the complex nature of discriminants, which are commonly used to determine the number of positive real roots of a polynomial (see Definition 3.4), more efficient estimation tools for the number of real roots are valuable to researchers working with polynomial systems. In this paper, we consider the reliability of specific estimation tools from tropical geometry.

To this end, we estimate the topology of the positive real root set, $\mathbb{Z}_+(f)$, for an arbitrary univariate trinomial by determining the structure of the positive tropical variety, denoted by $\text{Trop}_+(f)$. Through the use of Python, the accuracy of these estimates will be scrutinized based on exponent spread (see Definition 3.1) and coefficient variance. We will also define an upper bound on failure probabilty (see Definition 3.5) for all univariate trinomials.

In this paper, we will consider univariate trinomials, a simple instance of the n-variate, $n+2$-nomial, or *circuit* case. Our trinomials will be of the form $f(x) = c_1x^{\alpha_0} + c_2x^{\alpha_1} + c_3x^{\alpha_2}$, for $\alpha_0 < \alpha_1 < \alpha_2$, and c_i Gaussian random variables. Unless explicitly stated otherwise, we assume $\alpha_0 = 0$. For the rest of this paper, assume all trinomials f are of this form.

As proven in [1], the **Archimedean Tropical Variety** of a univariate polynomial f is

$$\{v \in \mathbb{R} \mid \max_{j\in\{1,\ldots,t\}} |c_j e^{a_j v}| \text{ is attained for at least two distinct values of } j\},$$

a definition which can be easily extended into higher dimensions.

We will denote as the **Positive Tropical Variety**

$$\text{Trop}_+(f) := \{v \in \mathbb{R} \mid \max_{j\in\{1,\dots,t\}} |c_j e^{a_j v}| \text{ is attained at some indices } j, j' \text{ with } c_j, c_j' < 0\}.$$

We finally conjecture that

Conjecture 1. *Where f is a trinomial with exponent spread s and a variance ratio for c_2 of σ, the failure probability of f can be upper bounded by a function of the form* $\min(O(\sqrt{s}\sigma),\ O(\sqrt{s}),\ O(\sqrt{s}\sigma^{-0.74}))$.

3 Definitions and Notation

Definition 3.1. *The* **spread** *of a trinomial f is* $\dfrac{\min(\alpha_1 - \alpha_0, \alpha_2 - \alpha_1)}{\alpha_2 - \alpha_0}$.

Note then that when α_1 is the midpoint of α_2 and α_0, the spread is maximized at 0.5. Similarly as α_1 approaches either α_0 or α_2, the spread approaches 0. In this way, the spread measures how far apart the exponents in our trinomial are.

Definition 3.2. *The* **support matrix** *of a trinomial f is* $A := \begin{bmatrix} 0 & a_1 & a_2 \end{bmatrix}$.

Definition 3.3. *The* **lifted support matrix** *of a trinomial f is* $\widehat{A} := \begin{bmatrix} 1 & 1 & 1 \\ 0 & a_1 & a_2 \end{bmatrix}$.

For any basis vector b for the right nullspace of $\widehat{A}$, we will denote as b_i the i^{th} coordinate of b. Note that the right nullspace of $\widehat{A}$ always has dimension 1.

Definition 3.4. *The* **discriminant** *of a trinomial f is* $\Delta_{\widehat{A}} := b_1 \log|c_1| + b_2 \log|c_2| + b_3 \log|c_3|$.

Recall that $\Delta_{\widehat{A}} > 0$ implies that the trinomial has two real roots. However, consider when $\text{sign}(b_i)$ and $\text{sign}(c_i)$ are either equal for all i or opposite for all i. (Note that in any other case the topology of Trop_+ is constant and isotopic to the positive real root set.) In these cases, then we knew that when $\Delta_{\widehat{A}} \leq b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2|$ then Trop_+ indicates that we have zero or one roots. Thus, Trop_+ does not effectively estimate the roots when $0 < \Delta_{\widehat{A}} \leq b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2|$.

Similarly, if $b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2| < 0$, then as for $b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2| \leq \Delta_{\widehat{A}}$ Trop_+ indicates that we have one or two roots. However, $\Delta_{\widehat{A}} < 0$ implies that the trinomial has zero positive real roots. Thus, here Trop_+ does not effectively estimate the roots when $b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2| \leq \Delta_{\widehat{A}} < 0$.

Definition 3.5. *The* ***failure region*** *of a trinomial f is, depending on the sign of b, either the region $0 < \Delta_{\widehat{A}} \leq b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2|$ or the region $b_1 \log|b_1| + b_2 \log|b_2| + b_2 \log|b_2| \leq \Delta_{\widehat{A}} < 0$ where $Trop_+$ fails to be isotopic to the positive real root set. The* **failure set** *of f is the set of coefficients $c = (c_1, c_2, c_3)$ such that* $\text{sign}(c) = \pm\, \text{sign}(b)$ *and $\Delta_{\widehat{A}}$ for f lies in the failure region. For f with fixed α_i, the* **failure probability** *of f is the probability that when (c_1, c_2, c_3) are randomly generated from a Gaussian distribution that c is in the failure set of f.*

4 Experimental Probability in Error Region

Our first experiments were examining the relationship between the spread of a trinomial f with its failure probability.

To estimate the failure probability we used 1,000,000 trails for each system of trinomials. In each trial we used Sage's Gaussian distribution with a standard deviation of 1 to generate variables c_1, c_2, and c_3, and then determined if the discriminant generated polynomial was in the failure region.

4.1 $f(x) = c_1 + c_2x + c_3x^2$

To start off with a simple case for the purpose of later comparisons, we began with the system of standard quadratic polynomials. For this system, we found the probability of lying within the failure region to be 5.9895%.

4.2 $f(x) = c_1 + c_2x^{26} + c_3x^{50}$

We chose to explore polynomials in this system since the exponents here have a large spread. For these polynomials, we found that the polynomial was in the failure region 5.9744% of the time.

4.3 $f = c_1 + c_2x^{99} + c_3x^{100}$

We chose to explore polynomials in this system to consider cases with a small spread, and found that these polynomials were in the failure region only 0.4471% of the time.

This led us to hypothesize that polynomials with a smaller ratio between the final two coefficients had a better failure probability (we noted from the exponent ratios in 3.1 and 3.2, for instance, that $\frac{1}{2} \approx \frac{26}{50}$ and 5.9895% $\approx$ 5.9744%).

4.4 $f = c_1 + c_2x^{19} + c_3x^{20}$

To test this hypothesis, we chose polynomials in this system, which has a small ratio though still a larger ratio than $\frac{99}{100}$, and indeed found that the failure probability was 1.6041%.

5 Exponent Ratio and Failure Probability

To test whether or not the failure probability is related to the ratio $\frac{\alpha_1}{\alpha_2}$ of a trinomial f, we explicitly experimented with how changing this ratio changed the failure probability.

5.1 Experimental Method

We first fixed $\alpha_2 = 100$, and let $\alpha_1 \in [1, 99]$, incrementing a_1 by steps of size 1. That is, we considered 99 ratios from 0.01 to 0.99. For each of the 100 ratios, we ran $1,000,000$ trials. Each trial we generated new random standard Gaussian coefficients for f and determined whether these coefficients were in the failure set of f. We finally found the failure probability across those $1,000,000$ trials.

5.2 Results

After running our experiments, we plotted the trials (Figure 1) and after examining the shape of the output, used scipy's `curve_fit` function to find a quadratic regression for the data. This gave us the regression

$$h(x) = 0.61353465 + 21.87751589x - 21.86653471x^2$$

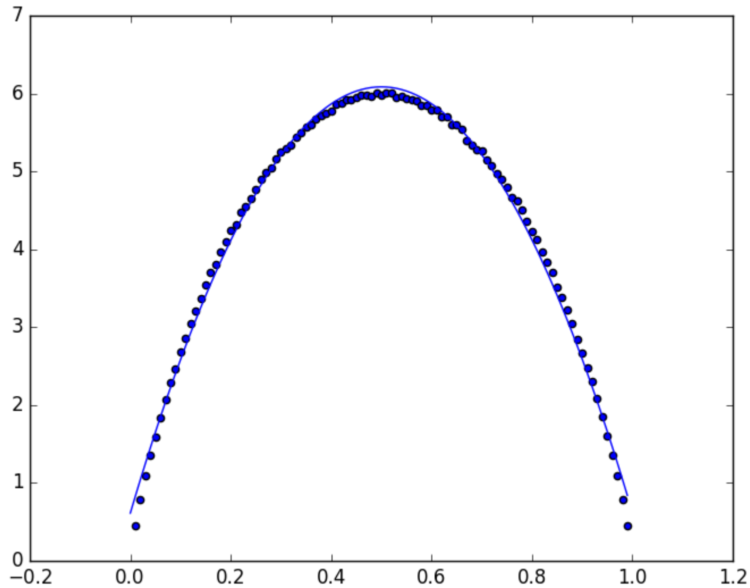

Figure 1: $\frac{\alpha_1}{\alpha_2}$ vs. Experimental Failure Probability

To see if the relationship changed with respect to the scale of the exponents involved, we then followed up by checking for $\alpha_2 = 25$ and $\alpha_1 \in [1, 24]$, in which case we again incremented α_1 by steps of size 1. These exponents gave us the similar regression

$$h(x) = 0.70553438 + 21.41546383x - 21.40520296x^2,$$

We also checked for $\alpha_2 = 1987$ and $\alpha_1 \in [19, 1900]$, incrementing by steps of size 19, which again resulted in a similar regression

$$h(x) = 0.65678311 + 21.55924563x - 21.47777145x^2.$$

5.3 Exponent Spread Experiment

As these were all polynomials with percentages that decreased as α_1 approached either 0 or α_2, and peaked when α_1 was roughly $\frac{\alpha_2}{2}$, we theorized that this curve depended not just on the ratio between α_1 and α_2 but actually the largest ratio between α_1 and either α_0 or α_2.

To test this, we looked at 100 trinomials of the form $f = c_1x^{24} + c_2x^{a_1} + c_3x^{626}$, $24 < a_1 < 626$, running 1,000,000 trials for each polynomial. This time graphing against the ratio between $\frac{24}{\alpha_1}$, we found a similar quadratic curve (Figure 2), with regression

$$h(x) = -0.27225719 + 23.51542209x - 21.77854389x^2$$

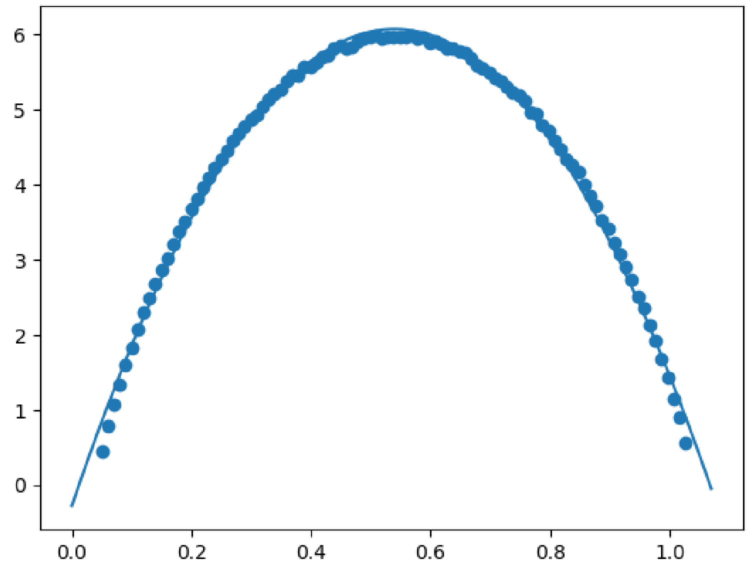

Figure 2: $\frac{24}{a_1}$ vs. Experimental Failure Probability

Noticeably, we still end up with a similar curve and regression, such that it is indeed clear that this relationship is dependent on the maximum ratio between α_1 and either α_0 or α_2, rather than simply the ratio between either of these specific pairs of exponents.

6 Variance Ratios and Failure Probability

Having examined the role of exponent ratio in shaping failure probability, the next step was to consider the affect of coefficient ratio on failure probability. Since we desired our coefficients to be Gaussian random variables, and thus could not set their exact ratio, we instead experimented on the ratio of the variances of the Gaussian distributions from which we generated the coefficients.

6.1 Trinomials with Large Spread

To test the effect of varying ratios between random Gaussian coefficients, we first examined trinomials of the form $f(x) = c_1 + c_2 x + c_3 x^2$.

Thus, $\widehat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\Delta_{\widehat{A}} = -\log|c_1| + 2\log|c_2| - \log|c_3|$. With these fixed values, we knew that our failure region was $0 < \Delta_{\widehat{A}} \leq \log|4|$.

6.1.1 Experimental Method

To study the effect of different variances on failure probability, we held the standard deviation of two of the coefficients constant, and changed the standard deviation of the last coefficient. We then examined how the ratio between these standard deviations affected the failure probability of f.

We first tested the effect of varying the variance of c_2. We generated c_1 and c_3 from standard Gaussian distributions while incrementing the standard deviation of the distribution for c_2, since numpy's Normal distribution relies on standard deviation rather than variance. We started with

$\sigma_2 = 0.1$ and increased by steps of 0.1. We first considered 100 standard deviations of $0.1 \leq \sigma_2 \leq 10$ with $1,000,000$ trials for each standard deviation. When we later varied σ_1 and σ_3, we ended up needing to test them over a wider range of values since the failure probability decreased at a slower rate.

In each trial, we first generated c_1 and c_3 from a standard Gaussian distribution. We then randomly generated c_2 from a Gaussian distribution with our current standard deviation. We then determined whether the polynomial lay in the failure region or not. After the $1,000,000$ trials, we calculated the probability that a polynomial with the current ratio of standard deviations would lie in the failure region.

6.1.2 Results

After running our experiments, we plotted the data (Figure 3) and after examining the shape of the output, used scipy's `curve_fit` function to fit various regressions to the data, eventually settling on a linear combination of regressions of the form xe^{-x} and $x^k e^{-x}$. This gave us the regression

$$h(x) = -1.03061413 + 15.572038x^{1.0356945}e^{-1.04617418x} + 1.76374323xe^{-0.20716401x}$$

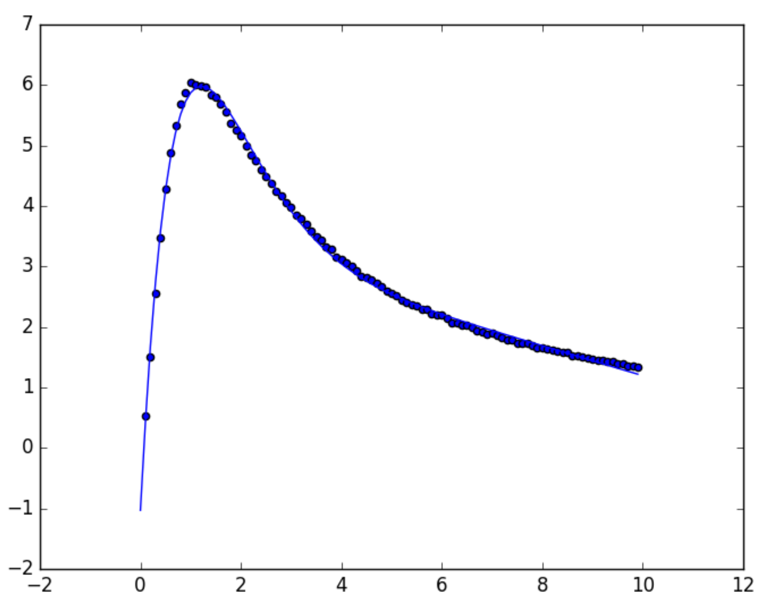

Figure 3: Quadratic c_2 Standard Deviation Ratio vs. Experimental Failure Probability

We then wanted to test if varying the standard deviation of c_3 would produce similar results. Graphing this (Figure 4), we get a similar regression fit to the linear combination of xe^{-x} and $x^k e^{-x}$

$$h(x) = 2.29455814 + 3.64985287x^{0.53354838}e^{-0.33694698x} + 7.19452508xe^{-1.85385613x}$$

When we altered the variance for the distribution for c_1, we ended up with a similar regression as for c_3,

$$h(x) = 2.33935321 + 3.68129608x^{0.55744044}e^{-0.35261467x} + 7.22056226xe^{-1.8871049x}$$

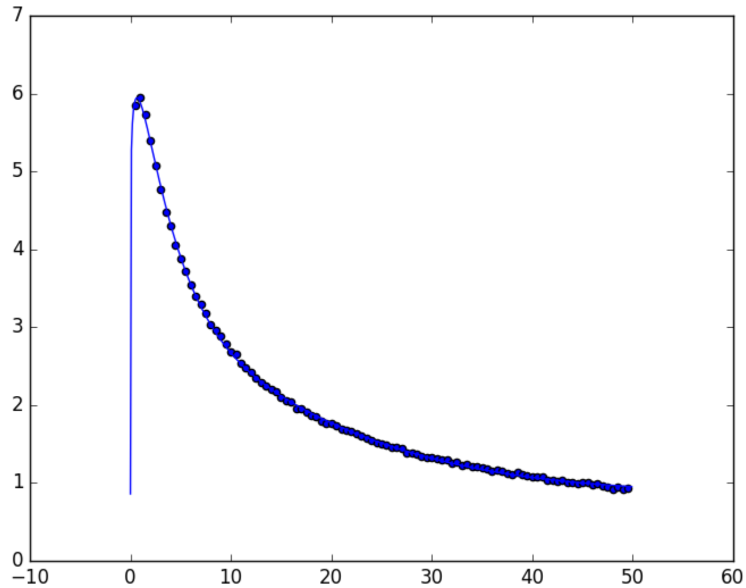

Figure 4: Quadratic c_3 Standard Deviation Ratio vs. Experimental Failure Probability

6.2 Trinomials with Small Spread

Our previous experiment of how variance ratios affected the failure probability used trinomials with a large exponent spread. As we saw in Section 5 that the spread of exponents affects the failure probability, here we consider for trinomials with a small exponent spread how the ratio of variances between random Gaussian coefficients affects the failure probability. Specifically, we consider trinomials in the system $f(x) = c_1 + c_2x^{99} + c_3x^{100}$.

Thus, $\widehat{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 99 & 100 \end{bmatrix}$ and $\Delta_{\widehat{A}} = -99\log|c_1| + 9900\log|c_2| - 9801\log|c_3|$. With these fixed values, we know that our failure region is $0 < \Delta_{\widehat{A}} \leq 554.415190113\ldots$.

6.2.1 Experimental Method

As described in Section 6.1.1, we first generated c_1 and c_3 from standard Gaussian distributions while incrementing the standard deviation of the distribution for c_2. We started with $\sigma_2 = 0.1$ and increased by steps of 0.1. We first considered 100 standard deviations of $0.1 \leq \sigma_2 \leq 10$ with $1,000,000$ trials for each standard deviation.

6.2.2 Results

After generating our data, we plotted the trials (Figure 5) and after examining the shape of the output, used scipy's `curve_fit` function to fit to a linear combination of xe^{-x} and x^ke^{-x}. This gave us the regression

$$h(x) = -0.06450709 + 0.18826155x^{0.55247034}e^{-0.15034146x} - 1.03096168xe^{-1.09906311x}$$

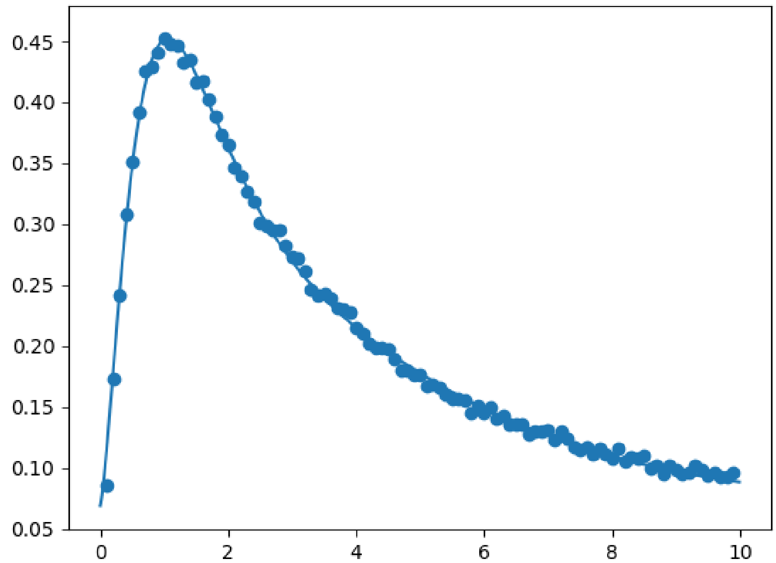

Figure 5: Small spread trinomial c_2 standard deviation ratio vs. experimental failure probability

We similarly tested the effect of varying the standard deviation of c_3. Plotting the data and using the `curve_fit` function gave us a similar regression,

$$h(x) = -0.04819254 + 0.17702176x^{0.63398958}e^{0.179153x} + 1.08224924xe^{-1.15326991x}$$

We finally tested the effect of varying the standard deviation for c_1, this time with only $1,000,000$ trials for each value of σ_1, and the same number and range of σ_1 values. Plotting the data clearly showed that changing the standard deviation here did not affect the probability of being in the failure region (Figure 6).

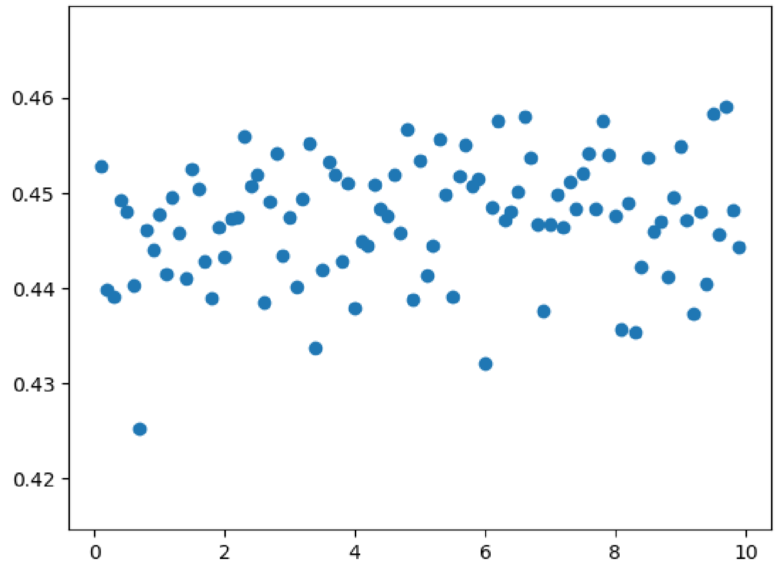

Figure 6: Small spread trinomial c_1 standard deviation ratio vs. experimental failure probability

7 Parameterizing Coefficients

Moving forwards from considering the relationship between the ratios of variance for the coefficients and the resulting probability of failure, our next goal was three-fold:

(a) We wished to simplify the fit functions for easier approximation,

(b) to see if we could modify these functions in order to transform them from approximations to bounding functions,

(c) and finally to see if we could express the coefficients of any bounding functions as a simple function of the exponent ratios.

That is, for trinomials of the form $f(x) = c_1 + c_2x^{\alpha_1} + c_3x^{\alpha_2}$, with $0 < \alpha_1 < \alpha_2$ and $c_1, c_3 \sim N(0,1)$ while $c_2 \sim N(0,\sigma)$, we wished to construct an upper bound on the failure probability as a function of σ. Further, we wished to find a relationship between this upper bound and $\max(\frac{\alpha_1}{\alpha_2}, \frac{\alpha_2-\alpha_1}{\alpha_2})$.

7.1 Piecewise Function Shape

Our first step towards this goal was to recognize that the variance regressions used in Section 6 could be simplified if, rather than trying to find a function that fit all the data, we instead expressed the function as a piece-wise function with much simpler components. In Figure 7, you can see that a linear fit line for σ before the maximum paired with a regression of the form σ^{-k} for σ after the maximum approximates the data well.

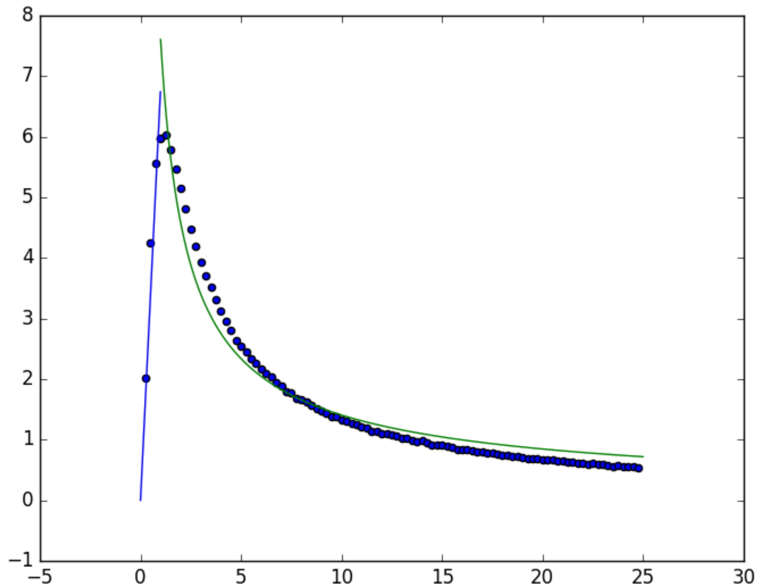

Figure 7: Piecewise linear and σ^{-k} fit functions for failure probability vs. σ

Thus we first needed to determine the σ value for the maximum failure probability, in order to know the domain for each function in the piecewise function. By considering exponent ratios from $[0.1, 0.9]$, we experimentally determined that the maximum always occurred at $\sigma = 1$. As such, we decided to ensure that the first linear function was an upper bound for $\sigma \in (0, 1]$ and that the second σ^{-k} function was an upper bound on the failure probability for $\sigma \in [1, \infty)$. As it was impossible to truly experiment for all $\sigma \in [1, \infty)$, we experimentally confirmed that σ^{-k} was an upper bound on the failure probability for all $\sigma \in [1, 100]$.

7.2 Linear Upper Bound: $\sigma \leq 1$

Our second concern was thus to construct a line through the origin that was an upper bound for all the points where $\sigma \leq 1$. To do so, we used the angles of all the experimental data points to calculate the exact minimum slope of such a line. In Figure 8, we compare such a line with the best-fit linear regression found by scipy's `curve_fit` function where our polynomial is the standard quadratic equation.

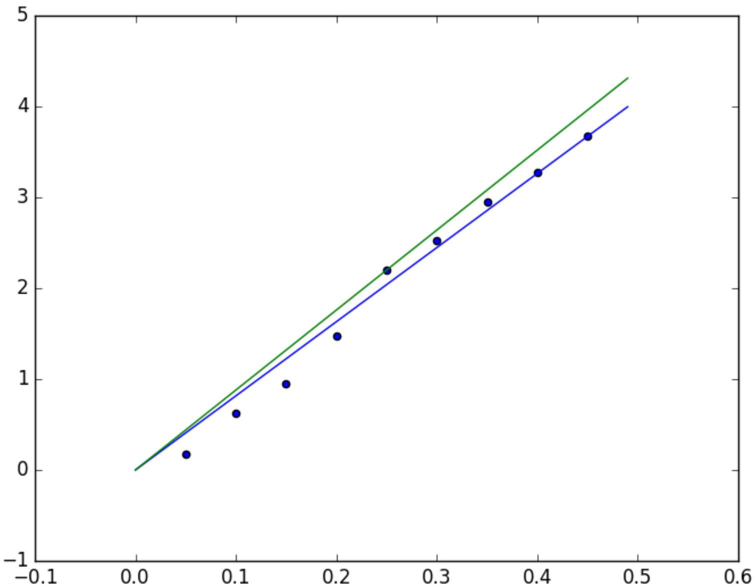

Figure 8: Linear upper bound and linear approximation for failure probability vs. $\sigma \leq 1$

After seeing through the quadratic example that a line of the form $g(\sigma) = a\sigma$ could indeed be used to upper-bound the failure probabilities for $0 < \sigma \leq 1$, we began to test how a varied with respect to $\frac{\alpha_1}{\alpha_2}$.

7.2.1 Experimental Method

We used 10 ratios of $\frac{\alpha_1}{\alpha_2} \in [0.1, 1]$. For each exponent ratio, we tested 10 different $\sigma \in [0.1, 1]$ with a step size of 0.1. For each σ we ran 100,000 trials, generating c_1 and c_3 from a standard Gaussian distribution and c_2 from a Gaussian distribution with standard deviation σ.

Once we had a sample distribution of failure probabilities with respect to standard deviation, we calculated a using angles as detailed in Section 7.2. For each σ, we ran 10 such experiments, and took the average of a over the experiments. We then plotted these a against the ratio of exponents for the polynomial, resulting in the distribution of minimum overline slopes seen in Figure 9.

As these slopes follow a clearly quadratic distribution, we also checked whether they had a possible relationship to $\sqrt{\frac{\max(\alpha_1, \alpha_2 - \alpha_1)}{\alpha_2}}$.

To do this, we now considered $g(\sigma) = C\sqrt{\frac{\max(\alpha_1, \alpha_2 - \alpha_1)}{\alpha_2}}\sigma$. This resulted in the linearized distribution of minimum overline slopes in Figure 9.

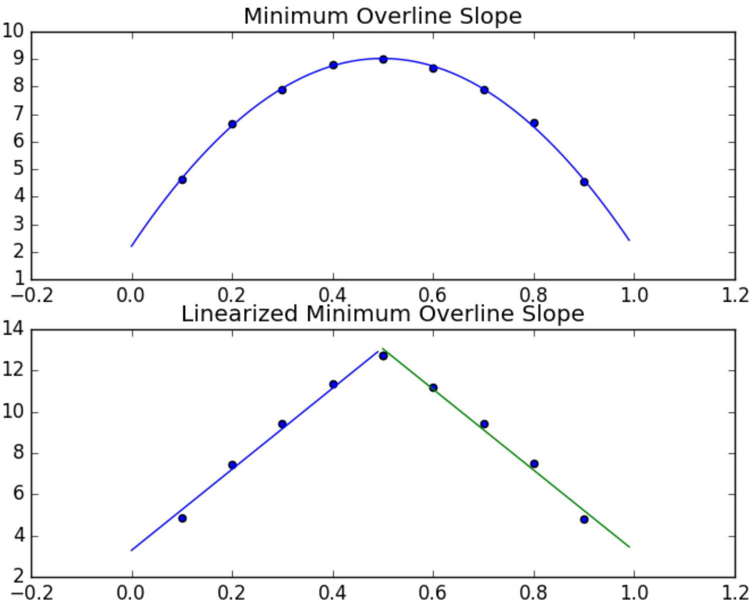

Figure 9: Minimum slopes for upper bound line vs. exponent ratio

Considering the linearized distribution in Figure 9, we found that the positive portion had slope 19.62435636 while the negative portion had slope 19.5743313. That is, if we let s denote the exponent spread, then $g(\sigma) = Cs\sqrt{1-s}\sigma$, for an appropriate C. This then suggests that for $\sigma \leq 1$, the failure probability can be bounded by a function of the form $O(\sqrt{s}\sigma)$.

7.3 σ^{-k} Upper Bound: $\sigma \geq 1$

Our next concern was to construct an upper bound for all $\sigma \geq 1$ of the form σ^{-k}. To do so, we used scipy's `curve_fit` function to fit a regression of the form $g(\sigma) = a\sigma^{-k}$ to experimentally generate data, and then incremented k until we had an upper bound curve. Once we knew where these minimal upper bound exponents were, we then worked to minimize a.

7.3.1 Experimental Method: finding k

We considered nine trinomials of the form $f(x) = c_1 + c_2x^{\alpha_1} + c_3x^{\alpha_2}$, where $c_1, c_3 \sim N(0,1)$ and $c_2 \sim N(0,\sigma)$, where $\frac{\alpha_1}{\alpha_2} \in [0.1, 0.9]$. For each trinomial we considered 100 values of $\sigma \in [1, 50]$. For each σ, we ran 100,000 trials to generate the failure probability for randomly generated c_1, c_2, c_3. We then generated a fit function and incremented k until it was an upper bound curve for the relationship between failure probability and σ. For each $\frac{\alpha_1}{\alpha_2}$ we averaged 10 such k and a values. We finally graphed the data points for a and k in relation to the exponent ratio, and could see that they both followed roughly quadratic distributions (Figure 10).

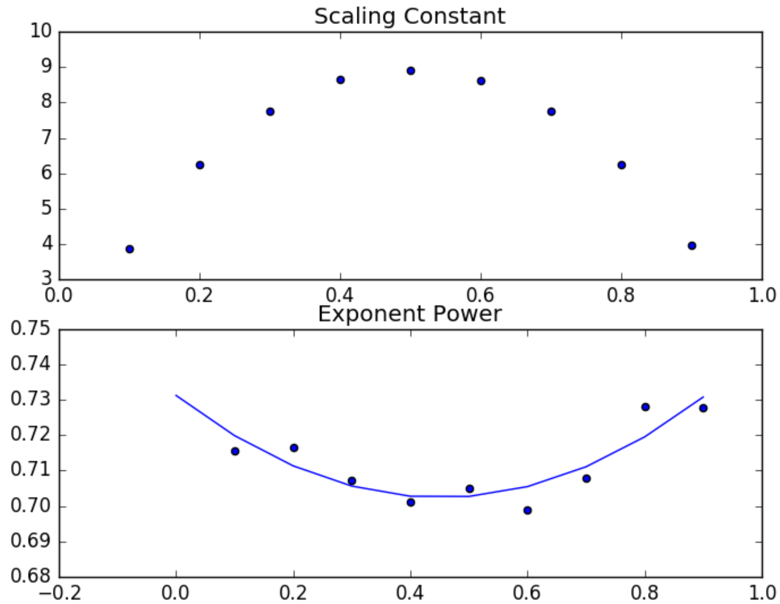

Figure 10: Minimum upper bound curve constants and exponents vs. polynomial exponent ratios

Since we had been trying to minimize k, and found that 0.74 was a upper bound on the values of k, we fixed $k = 0.74$.

7.3.2 Experimental Method: finding a

Having fixed k, we now wished to find a such that $g(\sigma) = a\sigma^{-0.74}$ upper bounded the failure probability for all exponent spreads.

We again considered nine trinomials with $\frac{\alpha_1}{\alpha_2} \in [0.1, 0.9]$, and 100 values of $\sigma \in [1, 50]$ for each. This time we ran 1,000,000 trials for each σ to generate the failure probability. We then found the minimum a to upper bound every data point for that exponent spread. Once we had done so, we let a be the maximum such across all exponent spreads, such that it upper bounded all.

7.3.3 Results

Through this process, we found that $g(\sigma) = 9\sigma^{-0.74}$ is an effective upper bound for our failure probabilities.

However, in plotting this upper bound line with our experimental data, it became clear that while this was a reasonably tight upper bound for the quadratic case (Figure 11), it was a much less tight upper bound for trinomials with a smaller exponent spread (Figure 12).

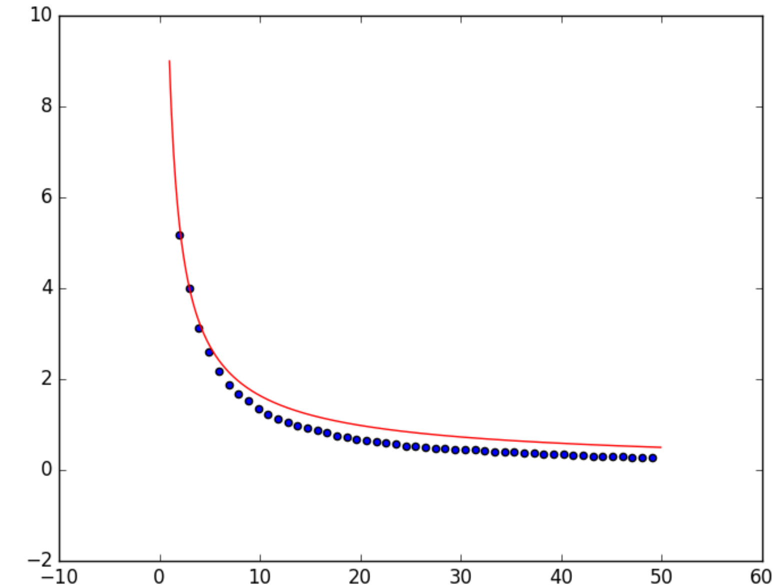

Figure 11: Upper bound for failure probability vs. σ for quadratic

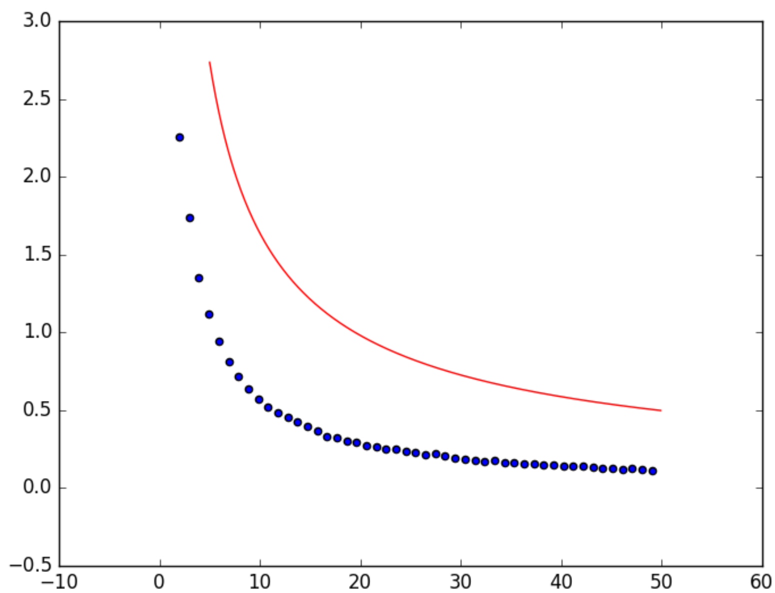

Figure 12: Upper bound for failure probability vs. σ for $A = (0, 9, 10)$

This suggests that in this case a might again have a dependence on the exponent ratio, leading us to experiment with whether we could linearize a here similarly to how we did in Section 7.2.1. That is, we considered $g(\sigma) = C\sqrt{\frac{\max(\alpha_1, \alpha_2 - \alpha_1)}{\alpha_2}}\sigma^{-0.74}$. We thus repeated our experiment to find the maximum a, but instead of finding the max across all exponent ratios, we graphed both a

and $C = a\sqrt{\dfrac{\alpha_2}{\max(\alpha_1, \alpha_2 - \alpha_1)}}$ for each ratio (Figure 13).

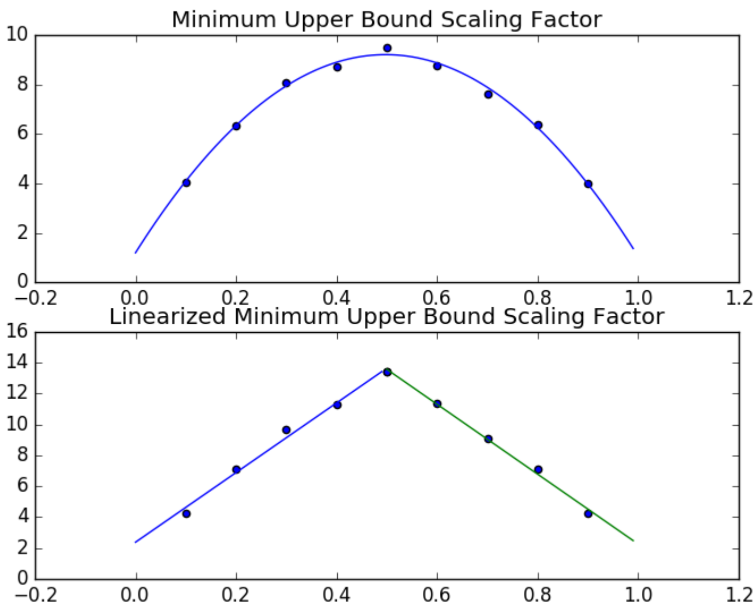

Figure 13: Constant factor for $a\sigma^{-k}$ vs. exponent ratio

Considering the linearized distribution in Figure 13, we found that the positive portion had slope 21.08217057 while the negative portion had slope 21.28604823. That is, if we let s denote the exponent spread, then $g(\sigma) = Cs\sqrt{1-s}\sigma^{-0.74}$, for an appropriate C. This then suggests that for $\sigma \geq 1$, the failure probability can be bounded by a function of the form $\mathrm{O}(\sqrt{s}\sigma^{-0.74})$.

It is noticeable that these experimental values for C are fairly close to those found in Section 7.2.1. Thus, whether these values are in fact related, or have some deeper meaning, is perhaps a question that further experiments or theory might be able to answer.

7.4 Constant Upper Bound: $\sigma \approx 1$

Since both of our upper bound lines are less accurate around $\sigma \approx 1$, both being much less tight of a bound around there, we also decided to add a third function to our piecewise function, where we simply estimate the maximum failure probability. That is, for $\sigma \approx 1$, we found upper bounds of the form $g(\sigma) = a$. Thus, our final piecewise function will choose the minimum failure probability bound from the three functions for any given σ.

7.4.1 Experimental Method: finding a

We considered nine trinomials of the form $f(x) = c_1 + c_2x^{\alpha_1} + c_3x^{\alpha_2}$, where $c_1, c_3 \sim N(0,1)$ and $c_2 \sim N(0,\sigma)$, where $\frac{\alpha_1}{\alpha_2} \in [0.1, 0.9]$. For each trinomial, recalling that the maximum occurred at $\sigma = 1$, we ran 1,000,000 trials to generate the failure probability for randomly generated c_1, c_2, c_3. For each $\frac{\alpha_1}{\alpha_2}$ we averaged 10 such failure probabilities a. We finally graphed the data points for a in relation to the exponent ratio, and could see that they followed roughly quadratic distributions (Figure 10).

Thus, as in Sections 7.2.1 and 7.3.3, we considered $g(\sigma) = C\sqrt{\frac{\max(\alpha_1, \alpha_2 - \alpha_1)}{\alpha_2}}\sigma$.

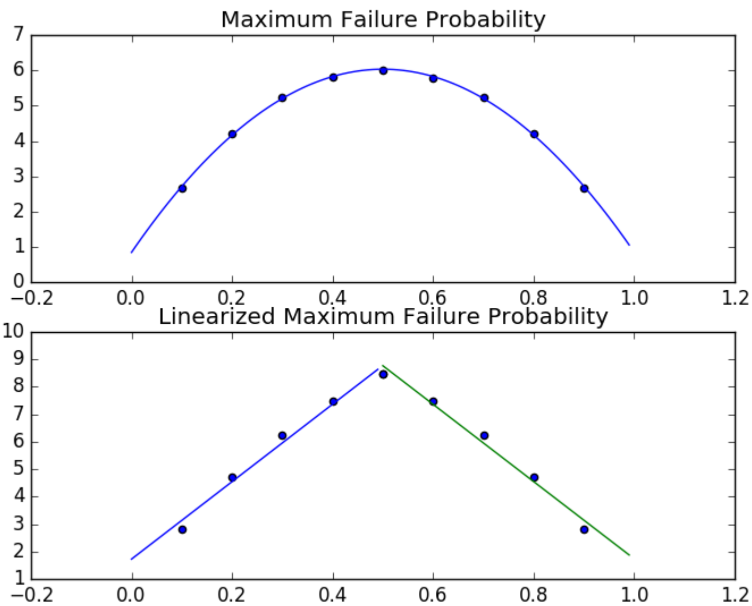

Figure 14: Maximum failure probabilities bound vs. polynomial exponent ratios

Considering the linearized distribution in Figure 14, we found that the positive portion had slope 14.09708497 while the negative portion had slope 14.05958356. That is, if we let s denote the exponent spread, then $g(\sigma) = Cs\sqrt{1-s}$, for an appropriate C. This then suggests that for $\sigma \approx 1$, the failure probability can be bounded by a function of the form $O(\sqrt{s})$.

8 Conclusion

In this paper we explored the probability that $\mathrm{Trop}_+(f)$ was not isotopic to $\mathbb{Z}_+(f)$. We considered the relationship between this probability and various characteristics of f, specifically the spread of the exponents of f and the ratio of the variances for f's coefficients.

Motivated by our experimental results relating the failure probability of f to its exponent and variance ratios, we conjectured that the failure probability can be upper bounded by a function of the form $O(\sqrt{s}\sigma)$ when $\sigma \leq 1$, by $O(\sqrt{s}\sigma^{-0.74})$ when $\sigma \geq 1$, and by $O(\sqrt{s})$ when $\sigma \approx 1$, where s is the exponent spread of f and σ is the variance ratio. That is, more simply, we conjecture that the failure probability of f can be estimated by a fucntion of the form $\min(O(\sqrt{s}\sigma), O(\sqrt{s}), O(\sqrt{s}\sigma^{-0.74}))$.

References

[1] M Avendano, R Kogan, M Nisse, and JM Rojas. Metric estimates and membership complexity for archimedean amoebae and tropical hypersurfaces, 2013. *Preprint.*