Characterizing Codes with Three Maximal Codewords

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Overview

Biological Motivation

- Encode spatial structure
- Associate neurons to regions of space
- Precisely fire in receptive fields

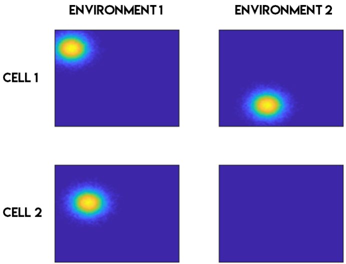


Figure: Neuron firing pattern

Biological Motivation Continued

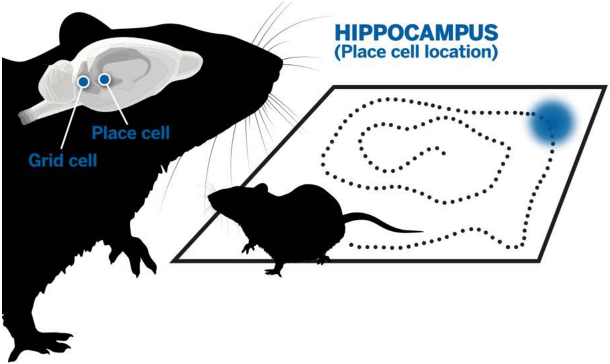


Figure: Place Cell Example

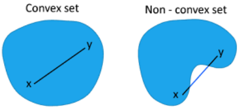
Definitions

Neural Code

- A *neural code* $\mathcal{C}$ on n neurons is a set of subsets of $[n]$ (called *codewords*), i.e. $\mathcal{C} \subseteq 2^{[n]}$.
- A maximal codeword in $\mathcal{C}$ is a codeword that is not properly contained in any other codeword in $\mathcal{C}$.
- *Convex* if it can be realized by a set of convex sets $U_1, U_2, ...U_n \subseteq \mathbb{R}^d$. A code's *minimal embedding dimension* is the smallest value of d for which this is possible.

Example

$\mathcal{C} = \{0, 1, 2, 3, 12, 23, 34, 13, 123, 234\}$, where $n = 4$.



Definitions Continued

Simplicial Complexes

An abstract *simplicial complex* on n vertices is a nonempty set of subsets (*faces*) of $[n]$ that is closed under taking subsets.

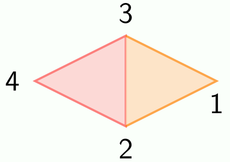
For a code $\mathcal{C}$ on n neurons, $\Delta(\mathcal{C})$ is the smallest simplicial complex on $[n]$ that contains $\mathcal{C}$:

$$\Delta(\mathcal{C}) := \{\omega \subseteq [n] \mid \omega \subseteq \sigma \text{ for some } \sigma \in \mathcal{C}\}.$$

Example

$\mathcal{C} = \{\emptyset, 1, 2, 3, 12, 23, 34, 13, 123, 234\}$

$\Delta(\mathcal{C}) = \{123,\ 234,\ 13,\ 34,\ 23,\ 12,\ 24,\ 4,\ 3,\ 2,\ 1,\ \emptyset\}$



Definitions Continued

Link

For a face $\sigma \in \Delta$, the *link of* σ *in* Δ is the simplicial complex

$$Lk_{\Delta}(\sigma) := \{\omega \subseteq \Delta \mid \sigma \cap \omega = \emptyset,\ \sigma \cup \omega \in \Delta\}.$$

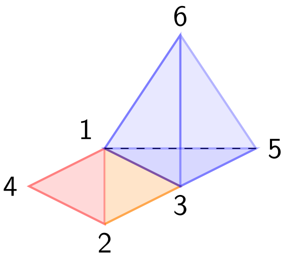


Figure: Δ

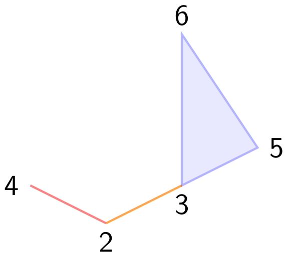
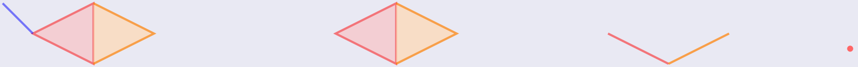


Figure: $Lk_{\Delta}(1)$

Definitions Continued

Contractible

A set is *contractible* if it can be reduced to one of its points by a continuous deformation.



Local Obstruction

If $Lk_{\Delta}(\sigma)$ is NOT contractible and $\sigma \notin \mathcal{C}$, a local obstruction occurs.

- σ is an intersection of maximal codewords.
- Local obstructions imply non-convexity

Definitions Continued

Max-intersection-complete

A code is *max-intersection-complete* if any arbitrary intersection of maximal codewords is in the original code.

- Max-intersection-complete $\Rightarrow$ convexity

Example

Max-intersection-complete code:

- $\mathcal{C} = \{123, 234, 145, 23, 4, 1\}$

Non max-intersection-complete code:

- $\mathcal{C} = \{123, 234, 145, 23\}$

Goal

Overarching Goal: Completely characterize codes with 3 maximal codewords

1. How to determine contractibility of triplewise intersections
2. Can we produce convex (open/closed) realizations for all codes
3. What are the embedding dimensions for the minimal/full codes

Contractibility

Lemma 4.7, (Curto et al.)

Let Δ be a simplicial complex. If $\sigma = \tau_1 \cap \tau_2$, where τ_1, τ_2 are distinct facets of Δ, and σ is not contained in any other facet of Δ, then the $Lk_\sigma(\Delta)$ is not contractible.

Thus, we only have to look at the triplewise intersection.

Case 1 - Link of Triplewise is Non-Contractible

- All other cases

Case 2 - Link of Triplewise is Contractible

- Triplewise intersection is non-empty and there are exactly 2 distinct pairwise intersections

Contractible

$\Delta(\mathcal{C}) = \{123, 124, 1356\}$ F_1, F_2, F_3

$\text{F}_1 \cap F_2 \cap F_3 = \{1\}$
$F_1 \cap F_2 = \{12\}$
$F_1 \cap F_3 = \{13\}$
$F_2 \cap F_3 = \{1\}$

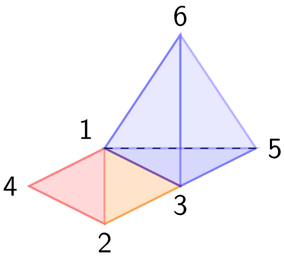


Figure: Δ

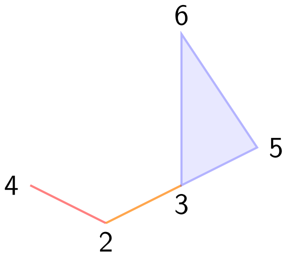


Figure: $Lk_{\Delta}(1)$

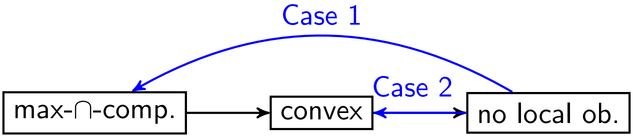
Convexity Relationship

Question: Does the absence of local obstructions imply convexity for codes with 3 maximal codewords?

Known Results

- max-intersection-complete $\Rightarrow$ convex $\Rightarrow$ no local obstructions
- max-intersection-complete $\not\Leftarrow$ convex ??? no local obstructions
  - $\not\Leftarrow$ for codes with 4 or more maximal codewords

Convexity Relationship



- Assume $\mathcal{C}$ has no local obstructions
  - Case 1: Non-contractible link
    - All intersections must be contained in $\mathcal{C}$, thus max-intersection-complete
  - Case 2: Contractible link
    - $\mathcal{C}$ is not required to be max-intersection-complete in order to have no local obstructions. Thus, we must provide a convex realization that such codes are indeed convex.
    - **Recall: contractible link if triplewise is nonempty & exactly 2 distinct pairwise**

Convexity Relationship

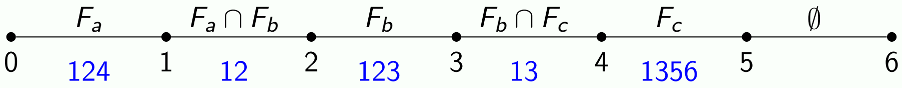
Minimal Code (2 distinct pairwise intersections)

A *minimal code* is the smallest code with no local obstructions.

Example: $\mathcal{C}_{min}(\Delta) = \{123, 124, 1356, 13, 12, 1\}$

Convex Realization for Case 2 Codes

Given a neural code $\mathcal{C}$ with three maximal codewords F_a, F_b, F_c such that $F_a \cap F_b \cap F_c = \sigma \neq \emptyset$, $F_a \cap F_b \neq \sigma$, $F_b \cap F_c \neq \sigma$ and $F_a \cap F_c = \sigma$. A convex (open/closed) realization of $\mathcal{C}_{min}(\Delta)$ can be constructed in dimension 1 such that the codewords appear in the following order:



Convexity Relationship

Question: Do no local obstructions imply convexity for codes with 3 maximal codewords?

Response: Yes. Assume $\mathcal{C}$ has no local obstructions.

1. Case 1 - Contractible: Convex Realization
2. Case 2 - Non-contractible: Max-$\cap$-complete

Convex Realizations

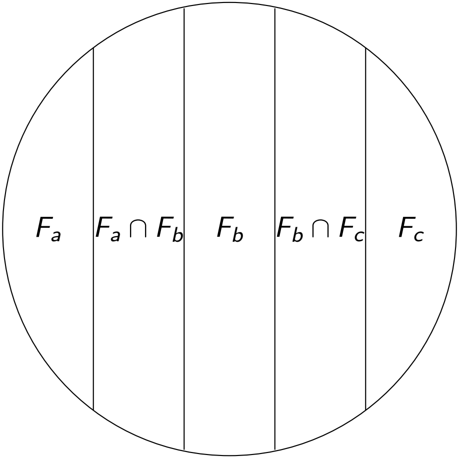


Figure: Realization of $\mathcal{C}_{min}(\Delta)$ in Dimension 2

Convex Realizations

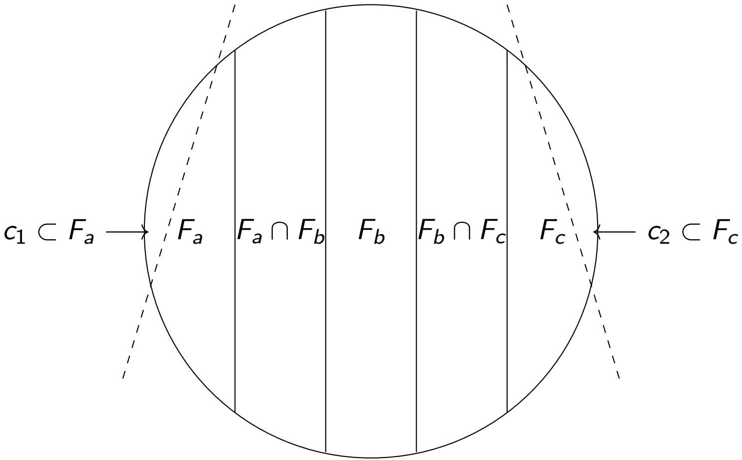


Figure: Realization of the code $\mathcal{C} = \{F_a, F_b, F_c, F_a \cap F_b, F_b \cap F_c, c_1, c_2\}$
$\mathcal{C}_{min}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$

Embedding Dimension

Embedding Dimension (Cruz et al.)

- For a minimal code, $\mathcal{C}_{min}(\Delta)$, consisting of only max codewords and their intersections, $\exists$ open/closed convex realization of $\mathcal{C}_{min}(\Delta)$ in $\mathbb{R}^{k-1}$, where k is the number of max codewords.
- Furthermore, by going to $\mathbb{R}^k$, you can get a realization of any code of the same simplicial complex that contains the minimal code.

Example

- $\mathcal{C}_{min}(\Delta) = \{123, 124, 1356, 13, 12, 1\}$ (Realizable in 2D)
- For a code, $\mathcal{C}$, such that $\mathcal{C}_{min}(\Delta) \subseteq \mathcal{C} \subseteq \Delta$ (Realizable in 3D)
  - $\mathcal{C} = \{\mathbf{123, 124, 1356, 13, 12, 1}, 2, 3, 4\}$

Embedding Dimension

- Expansion upon the result from Cruz et al:

Table: Minimal embedding dimension of $\mathcal{C}_{min}(\Delta)$ based on the number of pairwise intersections distinct from the triplewise

Embedding Dimension	**Pairwise Intersections**
1	0
1	1
1	2
2	3

Theorem 3.6 (Johnston - Spinner)

If $\mathcal{C}$ is a neural code with exactly 3 maximal codewords, then the minimal embedding dimension is at most 2.

Thank you for listening!

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