Calculating the Correlation Kernel along Space-Like Paths

Emily Bosche and Lauren Neudorf

July 29, 2021

Abstract

We study the representation theory that leads into Markov processes to culminate in the Markov process of the "push-block" model, which is defined by its interlacing property, as well as the space-like paths on which the particle system exists. Using previous work of Cerenzia '18 and Zhou '21, we define whether the Markov process of the symplectic group Sp_{2n} is a determinantal point process, and if it is, calculate the correlation kernel of such group along space-like paths.

1 Background

1.1 Representation Theory

Let a Lie Algebra g be a vector space with lie bracket $[\cdot,\cdot] : g \times g \to g$ such that g preserves the characteristics of skew-symmetry, bi-linearity, and the Jacobi identity. Let the representation φ of a lie algebra g be a homomorphism $\varphi : g \to g\ell(V)$ such that V is a finite vector space over $\mathbb{C}$. Let an irreducible representation be a nonzero representation with no subrepresentation.

Let Sp_{2n} be the sympletic lie group where

$$Sp_{2n} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

where A, B, C, and D are $n \times n$ matrices where $A = -D^T$, $B = B^T$, and $C = C^T$. The set of irreducible representations of Sp_{2n} is parametrized by
$\{(\lambda_1, ..., \lambda n) : \lambda_1 \geq ... \geq \lambda_n \geq 0\}$, for every $\lambda = (\lambda_1, ...\lambda_n)$ is mapped onto
$x = (x_1, ..., x_n)$, where $x_1 > ...n \geq 0$ and $x_i = \lambda_i + n - i$.

1.2 Markov Processes

In the case of a Markov chain, let X_t, $t \in \{0, 1, 2, ..., t\}$ denote a position at time t. Then, X_{t+1} only depends on X_t, not $\{X_{t-1}, ..., X_0\}$. A Markov process is then defined as the continuous time version of a Markov chain.

Definition 1.1 Over a discrete set $\mathbb{X} = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{+}$, if $P(x_1, ..., x_n) = \det[K(x_i, x_j]_{1 \leq i,j \leq n}$ for all $x_1, ..., x_n \in \mathbb{X}$ for some function K on $\mathbb{X} \times \mathbb{X}$, then the process is a determinantal point process.

Theorem 1.1 [Cerenzia '15, Cerenzia-Kuan '16] A Markov process at a fixed time t is determinantal.

1.3 The Push-Block Model

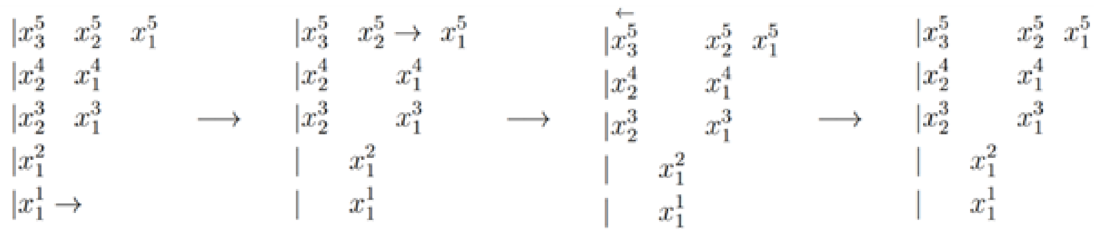

Figure 1: An illustration of the "push-block" model [Cerenzia '18]

The representation theory and Markov processes are joined together in the "push-block" model. Figure 1 illustrates the Markov process of a "push-block" model, where particles exist on a state space with reflecting barriers where $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{+}$. The state space that the particle system is on observes an interlacing property, where $X_{i+1}^{(K+1)} < X_i^{(K)} \leq X_i^{(K+1)}$ for odd values of K, and where $X_{i+1}^{(K+1)} \leq X_i^{(K)} < X_i^{(K+1)}$ for even values of K. The particles $X_n^{(2n)} < ... < X_1^{(2n)}$ come from representations of Sp_{2n} while the particles $X_n^{(2n-1)} < ... < X_1^{(2n-1)}$ come from odd symplectic groups. The heaviest particles lie at the bottom of the figure. Particles attempt to jump or push forward, but some are blocked from moving backwards as this growth continues by means of Markov processes.

2 The Problem

2.1 Open Problem

Write $K(\cdot,\cdot)$ for (x_i, n_i, t_i) for $1 \leq i \leq K$ and where $t_1 \leq ... \leq t_K$ and $n_1 \geq ... \geq n_K$.

2.2 Correlation Kernel

$K^{(t)}((s_1,n_1),(s_2,n_2)) = 1_{(n_1 \geq n_2)} \cdot \frac{2^{a_{n_1}+1/2}}{\pi} \int_{-1}^{1} J_{s_1,a_{n_1}}(x) J_{s_2,a_{n_2}}(x) \cdot (1-x)^{r_{n_1}-r_{n_1}+d_{r_1}}$

$(1+x)^{1/2} dx + \frac{2^{a_{n_1}+1/2}}{\pi} \int_{-1}^{1} \oint \frac{e^{t(x-1)}}{e^{t(x-1)}} J_{s_1,a_{n_1}}(x) J_{s_2,a_{n_2}}(u) \cdot \frac{(1-x)^{r_{n_1}+a_{n_1}}(1+x)^{1/2}}{(1-u)^{r_{n_1}}(x-u)} du dx$

2.3 Parameters

$(s_i, n_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_+$

$a_n = 1/2$ if n is even
$a_n = -1/2$ if n is odd

$r_n = \left\lfloor \frac{n+1}{2} \right\rfloor$, represents the number of particles on the n^{th} level

$J_{s,\pm 1/2}(x)$ represents a Jacobi polynomial

$\int_{-1}^{1} J_{s_1,\pm 1/2}(x) J_{s_2,\pm 1/2}(x)(1-x)^{\pm 1/2}(1+x)^{1/2} dx$

3 Our Progress

3.1 Cerenzia '18

Cerenzia's definitions provide the tools needed to find the correlation kernel at a given time. (2) provides Cerenzia's definition for τ at level n, a given that time $t_i^{2n-1/2+a} < t_j^{2n-1/2+a}$, while (3) provides his definition for ϕ, where $t_{b_1}^{2n_1-1/2+a_1} > t_{b_2}^{2n_2-1/2+a_2}$ and where $n_1 \leq n_2$.

$$\tau^{n,a}_{t_j^{2n-1/2+a}, t_i^{2n-1/2+a}} = \tau^{n,a}_{t_i^{2n-1/2+a}, t_{i-1}^{2n-1/2+a}} * \dots * \tau^{n,a}_{t_j^{2n-1/2+a}, t_{j-1}^{2n-1/2+a}} \tag{2}$$

$$\phi^{t_{b_1}^{2n_1-1/2+a_1}, t_{b_2}^{2n_2-1/2+a_2}} = \tau^{n_2,a_2}_{t^{2n_2+a_2+1/2}_{c(2n_2+a_2+1/2)}, t^{2n_2}_{b_2}} * \phi^{n_2,a_2}_{y} * \tau^{y}_{t^{y}_{c(y)}, t_0} * \dots * \phi^{m}_{n_1,a_1} * \tau^{n_1,a_1}_{t^{2n_1+a_1+1/2}_{b_1}, t^{n_1,a_1}_{0}} \tag{3}$$

In (3), y is the level below n_2, a_2 where $y = n_2, -1/2$ if $a_2 = 1/2$, while $y = n_2 - 1, 1/2$ if $a_2 = -1/2$. In addition, m is the level above n_1, a_1, where $m = n_1 + 1, -1/2$ if $a_1 = 1/2$, while $m = n_1, 1/2$ if $a_1 = -1/2$.

3.2 Zhou '21

Denote the particle positions at time t by $X_K^{n,a}$, let $c(i)$ denote arbitrary integers, and let $a = \pm 1/2$. Let $\phi^{N,+}_{N,-}(\cdot,\cdot)$ and $\phi^{N,-}_{N-1,+}(\cdot,\cdot)$ be functions, and let $\tau^{K,\pm}_{s,t}(\cdot,\cdot)$ be a function where the time depends on the level. In addition, we have times t, where

$$
\begin{aligned}
&0 = t_0^{2N-1/2+a} \leq \dots \leq t^{2N-1/2+a}_{c(2N-1/2+a)} \\
&0 = t_0^{2N-1/2+a-1} \leq \dots \leq t^{2N-1/2+a-1}_{c(2N-1/2+a-1)} \\
&0 = t_0^{2N-1/2+a-2} \leq \dots \leq t^{2N-1/2+a-2}_{c(2N-1/2+a-2)} \\
&\vdots \\
&0 = t_0^1 \leq \dots \leq t^1_{c(1)}
\end{aligned}
$$

Theorem 2.1 [Zhou '21] If $a = -1/2$ and the probability is of the form

$$const \times \prod_{n=1}^{N} [det[\phi^{n,-}_{n-1,+}(x_\ell^{n,-}(t^{2n-1}_{c(2n-1)}), x_K^{n-1,+}(t_0^{2n-2}))]_{1\leq K,\ell\leq n}$$

$$\times \prod_{b=1}^{c(2n-1)} det[\tau^{n,-}_{t_b^{2n-1}, t_{b-1}^{2n-1}}(x_\ell^{n,-}(t_b^{2n-1}), x_K^{n,-}(t_{b-1}^{2n-1}))]_{1\leq K,\ell\leq n}$$

$$\times det[\phi^{n,+}_{n,-}(x_\ell^{n,+}(t^{2n}_{c(2n)}), x_K^{n,-}(t_0^{2n-1}))]_{1\leq K,\ell\leq n}$$

$$\times \prod_{b=1}^{c(2n)} det[\tau^{n,+}_{t_b^{2n}, t_{b-1}^{2n}}(x_\ell^{n,+}(t_b^{2n}), x_K^{n,+}(t_{b-1}^{2n}))]_{1\leq K,\ell\leq n}]$$

$$\times det[\Psi^{N,a}_{N-\ell}(x_K^{2N-1/2+a}(t_0^{2N-1/2+a}))]_{1\leq K,\ell\leq N},$$

then the final determinant of $\phi^{N,+}_{N,-}$ does not appear and neither does the product of determinants of $\tau^{N,+}$, and so the process is determinantal.

3.3 Work of Cerenzia '18 in Framework of Zhou '21

In order to calculate the correlation kernel, we combine the definitions from Cerenzia's work into the framework of Zhou, so that we can calculate the kernel along space-like paths rather than at a fixed time. Such work resulted in

$$const \times \prod_{n=1}^{N}[det[\phi_{n_1-1,+}^{n_1,-}(x_\ell^{n,-}(t_j^{2n_1-1}), x_K^{n_1-1,+}(t_i^{2n_1-1}))]_{1\leq K,\ell\leq n}$$

$$\times \prod_{b_1=1}^{c(2n_1-1)} det[\tau^{n_1,-}_{t_{b_1}^{2n_1-1}, t_{b_1-1}^{2n_1-1}}(x_\ell^{n_1,-}(t_{b_1}^{2n_1-1}), x_K^{n_1,-}(t_{b_1-1}^{2n_1-1}))]_{1\leq K,\ell\leq n}$$

$$\times det[\phi_{n_2,-}^{n_2,+}(x_\ell^{n_2,+}(t_j^{2n_2}), x_K^{n_2,-}(t_i^{2n_2-1}))]_{1\leq K,\ell\leq n}$$

$$\times \prod_{b_2=1}^{c(2n_2)} det[\tau^{n_2,+}_{t_{b_2}^{2n_2}, t_{b_2-1}^{2n_2}}(x_\ell^{n_2,+}(t_{b_2}^{2n_2}), x_K^{n_2,+}(t_{b_2-1}^{2n_2}))]_{1\leq K,\ell\leq n}]$$

$$\times det[\Psi_{N-\ell}^{N,a}(x_K^{2N-1/2+a}(t_i^{2N-1/2+a}))]_{1\leq K,\ell\leq N}$$

3.4 Equation 33 [Cerenzia '18]

Cerenzia provides the definition for an inner product in respect to a normalized weight as

$$\langle f, g\rangle_a := \frac{2^{a+(1/2)}}{\pi}\int_{\mathbb{R}} f(x)g(x)w_{(a,1/2)}(x)dx \tag{4}$$

In applying (4) to our work, we used a generalized form of the τ terms from subsection 3.2. We renamed $\tau^{n,-}_{t_b^{2n-1}, t_{b-1}^{2n-1}}$ and $\tau^{n,+}_{t_b^{2n}, t_{b-1}^{2n}}$ with the generalized form of $\tau^{n,a}_{t_1,t_2}(x,y)$. We then used the following definition

$$\tau^{n,a}_{t_1,t_2}(x,y) = \left\langle J_{x,a}, J_{y,a}\varphi^{t_1-t_2}\right\rangle_a \tag{5}$$

where $f = J_{x,a}$, $g = J_{y,a}\varphi^{t_1-t_2}$, and $\varphi^{t_1,t_2} \to \varphi^t(x) = e^{t(x-1)}$, in combination with (4) to achieve the following result.

$$\frac{2^{a+1/2}}{\pi}\int_{\mathbb{R}} J_{x,a}(x)J_{y,a}e^{t(x-1)}(x)w_{(a,1/2)}(x)dx \tag{6}$$

3.5 Equation 34 [Cerenzia '18]

Though the following equation was not part of our completed work, we include the following definition as its intended use will be addressed in our final discussion.

$$T(x) = \sum_{k=0}^{\infty} \langle J_{k,a_n}, T\rangle_{a_n} J_{k,a_n}(x) \tag{7}$$

3.6 Lemma 2.2 [Borodin-Kuan '11]

In order to simplify the integration in Equation 33 [Cerenzia '18] (6), Lemma 2.2 is required. In [Borodin-Kuan '11], Lemma 2.2 is stated and proved for $a = \pm 1/2$, $b = -1/2$, $-1 \leq \zeta \leq 1$, with Test Function T∈C_1[-1,1], then:

$$T(\zeta) = \sum_{k=0}^{\infty} \int_{-1}^{1} \frac{J_k^{a,-1/2}(x)J_k^{a,-1/2}(\zeta)}{h_k^{a,-1/2}} T(x)(1-x)^a(1+x)^{-1/2}dx \tag{8}$$

Proving this lemma will require two equations from background text:

$$h_k^{(a,b)} = \frac{\pi c_k^2}{W^{(a,b)}(k)} \tag{9}$$

Where $W^{(a,b)}(k)$ is equal to 0 if $k > 0$ and $a = b = -1/2$, 1 if $k = 0$ and $a = b = -1/2$, and 1 if $k \leq 0$, $a = 1/2$ and $b = -1/2$. [Borodin-Kuan '11]. The next two equations are from [Szegö, '75]:

$$P_n^{(-1/2,-1/2)}(x) = \frac{1\times 3\times 5 ... (2n-1)}{2\times 4\times 6 ... 2n} T_n(x) = \frac{1\times 3\times 5 ... (2n-1)}{2\times 4\times 6 ... 2n} \cos(n\theta) \tag{10}$$

The next equation needed is for the mixed variable case:

$$P_n^{(1/2,1/2)}(x) = 2\frac{1\times 3\times 5 ... (2n+1)}{2\times 4\times 6 ... (2n+2)} \frac{\sin(\theta(n+1))}{\sin\theta} \tag{11}$$

Where $P_n^{(a,-1/2)}(x)$ is equal to our variable $J_k^{a,-1/2}(x)$ representing a Jacobi Polynomial where both a and b equal -1/2 in (10) and is a mixed case in (11). Combining these two equations for a case where $a = b = -1/2$, letting $x = \cos\phi$ and $\zeta = \cos\theta$,

$$\frac{J_k^{(-1/2,1/2)}(x)J_k^{(-1/2,1/2)(\zeta)}}{h_k^{(-1/2,1/2)}} = \frac{1}{\pi} if k = 0 \frac{2}{\pi}\cos(k\phi)\cos(k\theta)\ if k > 2 \tag{12}$$

Since T is C^1, the Fourier series of T converges to T:

$$T(\cos(\phi)) = \hat{T}_0 + \hat{T}_1 \cos(\phi) + \hat{T}_2 \cos(2\phi) + ..., \tag{13}$$

Where

$$if k = 0, \hat{T}_k = \frac{1}{\pi} \int_0^{\pi} T(\cos(\phi)) d\phi, \tag{14}$$

$$if k > 0, \hat{T}_k = \frac{2}{\pi} \int_0^{\pi} T(\cos(\phi) \cos(k\phi) d\phi \tag{15}$$

Therefore,

$$\sum_{k=0}^{\infty} \int_{-1}^{1} \frac{J_k^{a_1-1/2}(x) J_k^{a_1-1/2}(\zeta)}{h_k^{a_1-1/2}} T(x)(1-x)^{-1/2}(1+x)^{-1/2} dx \tag{16}$$

$$= \frac{1}{\pi} \int_0^{\pi} T(\cos\phi) d\phi + \frac{2}{\pi} \sum_{k=1}^{\infty} \int_0^{\pi} T(\cos\phi) \cos(k\phi) \cos(k\theta) d\theta \tag{17}$$

$$= \hat{T}_0 + \hat{T}_1 \cos(\phi) + \hat{T}_2 \cos(2\phi) + ..., = T(\cos\phi) = T(\zeta) \tag{18}$$

For the case where a=1/2, the mixed Jacobi Polynomial from [Szegö, '75] would be utilized, and

$$\frac{J_k^{(1/2,-1/2)}(x) J_k^{(1/2,-1/2)}(\zeta)}{h_k^{(1/2,-1/2)}} = \frac{\sin((k+1/2)\phi)}{\sin(\phi/2)} \frac{\sin(\theta(k+1/2))}{\sin(\theta/2)} \tag{19}$$

Then the rest of this part of the proof would follow similarly to the previous case.

3.7 Analog of Lemma 2.2

However, our problem mainly focuses on the case where $b = 1/2$, and thus we will need to follow a similar outline for the proof of the analog of Lemma 2.2, where $a = \pm 1/2$, and $b = 1/2$, with slight variations. Instead of expanding T using a cosine fourier series with a basis of $\cos(kx)$, we will use $\sin(kx)$ instead to prove our first case (non-mixed), and apply said expansion into the expression $\frac{J_k^{(a,1/2)}(x) J_k^{(a,1/2)}(\zeta)}{h_k^{(a,1/2)}}$. To accomplish this, we will need to know the values of the mixed and non-mixed Jacobi Polynomials from [Szegö, '75], and the value of $h_k^{(a,1/2)}$, which is the same formula stated in 3.6. For $a = b = 1/2$, $W^{(a,b)}(k) = 2$, and for $a = -1/2$, $b = 1/2$, $W^{(a,b)}(k) = 1$. With the non-mixed and mixed case polynomial equations as follows:

$$P_n^{(1/2,1/2)}(x) = 2 \frac{1 \times 3 \times 5...(2n+1)}{2 \times 4 \times 6...(2n+2)} \frac{\sin(\phi(n+1))}{\sin\phi} \tag{20}$$

$$P_n^{(-1/2,1/2)}(x) = \frac{1 \times 3 \times 5...(2n-1)}{2 \times 4 \times 6...2n} \frac{\cos((2n+1)(\phi/2))}{\cos((\phi/2))} \tag{21}$$

Utilizing both these equations for $a = 1/2$, and again setting $x = \cos\phi$ and $\zeta = \cos\theta$,

$$\frac{J_k^{(1/2,1/2)}(x)J_k^{(1/2,1/2)(\zeta)}}{h_k^{(1/2,1/2)}} = (\frac{2}{\pi})(\frac{\sin((k+1)\phi)}{\sin(\phi)})(\frac{\sin((k+1)\theta)}{\sin(\theta)}) \tag{22}$$

With weight,

$$w = (1-x)^{1/2}(1+x)^{1/2} = \sin\phi \tag{23}$$

Now expanding T:

$$T(\cos(\phi)) = \sum_{k=0}^{\infty} \hat{T}_k \frac{\sin(k\phi)}{\sin\phi} = \hat{T}_0 + \hat{T}_1 \frac{\sin(2\phi)}{\sin\phi} + ..., \tag{24}$$

Where

$$\hat{T}_k = \frac{2}{\pi}\int_0^{\pi} T(\cos\phi)\frac{\sin((k+1)\phi)}{\sin\phi}d\phi \tag{25}$$

Therefore

$$\sum_{k=0}^{\infty}\int_{-1}^{1}\frac{J_k^{1/2,1/2}(x)J_k^{1/2,1/2}(\zeta)}{h_k^{1/2,1/2}}T(x)(1-x)^{1/2}(1+x)^{1/2}dx \tag{26}$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}(\frac{\sin((k+1)\theta)}{\sin\theta}\int_0^{\pi} T(\cos\phi)\frac{\sin((k+1)\phi)}{\sin\phi}d\phi) \tag{27}$$

$$= \hat{T}_0 + \hat{T}_1\frac{\sin(2\theta)}{\sin\theta} + ..., = T(\cos\theta) = T(\zeta) \tag{28}$$

Thus proving the case when $a = b = 1/2$.
When $a = -1/2$, $b = 1/2$,

$$\frac{J_k^{(-1/2,1/2)}(x)J_k^{(-1/2,1/2)}(\zeta)}{h_k^{(-1/2,1/2)}} = \frac{1}{\pi}\frac{\cos((2n+1)(\phi/2))}{\cos((\phi/2))}\frac{\cos((2n+1)(\theta/2))}{\cos((\theta/2))} \tag{29}$$

Utilizing this expression and a similar method used with previous cases, the rest of the proof for this case follows.

We are able to use this analog of Lemma 2.2 to simplify the integration in (6) and properly calculate convolutions of ϕ and t. We can then use our derived equations to find these solutions along space-like paths.

3.8 Proposition 4.2

Proposition 4.2 from [Cerenzia '18] will assist us later on in calculating values of ϕ and Ψ. For any $(s,n),(t,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$, define the functions:

$$\Phi^m_{r_m-k}(t) := \frac{1}{2\pi i}\oint \frac{J_{t,\alpha_m}(w)}{E(w)(w-1)^{r_m-k+1}dw} \tag{30}$$

$$\phi^{[n,m)}(s,t) := -\frac{1}{2\pi i}\oint \left\langle J_{s,\alpha_n}, \frac{J_{t,\alpha_m}(u)(u-1)^{r_n-r_m}}{x-u} \right\rangle_{\alpha_n} du, n < m \tag{31}$$

Where the contours are positively oriented simple loops around the interval [-1,1] and contain no zeros of E. If these conditions are met, then the simple point process $\tilde{\chi^w}$, which is determined by the push-forward measure $\tilde{\xi^w}$ of equation [50] in [Cerenzia '18], has a determinantal correlation function $\tilde{\rho^w}$ with kernel:

$$K^w((s,n),(t,m)) = -\Phi^{[n,m)}(s,t)1_{(n<m)} + \sum_{k=1}^{r_m} \Psi^n_{r_n-k}(s)\Phi^m_{r_m-t}(t) \tag{32}$$

$\tilde{\xi^w}$ is also determinantal and follows the same conditions as (10).

4 Discussion

Unfortunately, we were unable to come to final results throughout the project. The following steps have left to be completed. First, using the proof of Lemma 2.2 where $a = -1/2$, the result in (6) must be simplified, in which case, (7) would be used to calculate a summation of polynomials, $T(x)$, which would be used in explicitly calculating the correlation kernel. Next, Using Proposition 4.2, values for ϕ and ψ must be calculated. Using such ϕ, ψ, and T values, $det[\psi]det[T]det[\phi]$ can be determined. If such product is equal to the probability of the form in Theorem 2.1, and given that (s_i, n_i) is occupied at time t_i for $1 \leq i \leq k$, then we can conclude that there is an explicit formula for K, which is the correlation kernel itself. From this point, we would be able to explicitly calculate such correlation kernel.

Acknowledgements

We especially thank our advisor, Dr. Jeffrey Kuan, and our TA, Zhengye Zhou, for their support and guidance through this summer's Research Experience for Undergraduates. We greatly extend this appreciation to Texas A&M University and the NSF-funded REU (DMS-1757872) for an incredible opportunity and experience this summer.

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