On Property P_1 and Spaces of Operators

Stephen Rowe

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Operator Spaces and Algebras

A space of operators in finite dimensions is a set of matrices that is a vector space. That is, it is closed under addition and scalar multiplication.

An algebra of operators is a space of operators that is also closed under multiplication. That is, if X is our space and $a, b \in S$, then $ab \in S$

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$X_\perp = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \right\}$

Property P_1

A space of operators, $S \subseteq M_n(\mathbb{C})$, is said to have property P_1 if every element $M_n(\mathbb{C})$ can be written as the sum of an element of the preannihilator and a rank-1 matrix.

$M_n(\mathbb{C}) = S_\perp + R_1$

For any subspace of $M_n(\mathbb{C})$, we can write $M_n(\mathbb{C}) = S_\perp + S^*$.

Therefore, to check if S has property P_1, we only need to check if $S_\perp + R_1 = S^*$.

Example

$$\text{Let } S = \left\{ \begin{pmatrix} 0 & a & b & c \\ d & 0 & 0 & 0 \\ e & 0 & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix} \right\} = S^* \subset M_4(\mathbb{C})$$

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$$S_\perp = \left\{ \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & x_{32} & x_{33} & x_{34} \\ 0 & x_{42} & x_{43} & x_{44} \end{pmatrix} \right\}$$

We need to show $S^* = S_\perp + R_1$, or alternately given any $t \in S$, there exists a $t_\perp \in S_\perp$ such that $t + t_\perp$ is rank-1 for some .

Example Continued

$$t + t_{\perp} = \left\{ \begin{pmatrix} x_{11} & a & b & c \\ d & x_{22} & x_{23} & x_{24} \\ e & x_{32} & x_{33} & x_{34} \\ f & x_{42} & x_{43} & x_{44} \end{pmatrix} \right\}$$

Separating Vectors

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Example: $\begin{pmatrix} a & 0 & b \\ 0 & b & c \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + bz \\ by + cz \\ cz \end{pmatrix} = 0$

Separating Vector Results

Let $S \subseteq M_n(\mathbb{C})$. If S has a separating vector, then S has property P_1. This provides a quick way of showing a space has P_1.

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If $dimS > n$, then S cannot have a separating vector.
Let $A = \oplus_{i=1}^{m} A_i$. If each A_i has a separating vector, then A has a separating vector.

Spaces Generated by Two Operators

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Basic Properties

If a space S has property P_1 and T is a subspace of S, then T also has property P_1.

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If S has property P_1, then so does S^*.

More Properties

If $S \subset M_n(\mathbb{C})$ is a space with property P_1 and $a, b \in M_n(\mathbb{C})$ are invertible operators, then the space aSb also has property P_1.

If $A \subset M_n(\mathbb{C})$ is an algebra with property P_1 and $p \in M_n(\mathbb{C})$, then pAp also has property P_1.

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If $S \subset M_n(\mathbb{C})$ is a space with property P_1 and $a, b \in M_n(\mathbb{C})$ are invertible operators, then the space aSb also has property P_1.
If $A \subset M_n(\mathbb{C})$ is an algebra with property P_1 and $p \in M_n(\mathbb{C})$, then pAp also has property P_1. Let $T \in M_n(\mathbb{C})$ and $W(T) = [I, T, T^2, T^3, ...]$. This space has property P_1.

Maximum Dimension

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Ampliations

The 2-ampliation of a space S is the a new space $S^{(2)} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$.

Similarly, the n-ampliation of S is the space

$$S^{(n)} = \begin{pmatrix} S & 0 \dots & \\ & \ddots & \\ 0 & \dots & S \end{pmatrix}$$

Let $A = M_n(\mathbb{C})$. Then, $S^{(n)}$ has property P_1 because it has a separating vector. The separating vector can be constructed as $\bigoplus_{i=1}^{n} e_i$.

Semi-Simple Algebra

A semi-simple algebra $A \subset M_n(\mathbb{C})$ is an algebra that can be written as the direct sum of full matrix algebras ampliated to their respective dimension. That is, $A = \oplus_{i=1}^{k} M_{n_i}^{(n_i)}(\mathbb{C})$.

Example: $\mathbb{C} \oplus M_2^{(2)}(\mathbb{C}) = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \end{pmatrix}$

Semi-Simple Algebras have P_1

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Let B be a semi simple algebra. Then, $B = \oplus_{i=1}^{k} M_{n_i}^{(n_i)}(\mathbb{C})$. Each $M_{n_i}^{(n_i)}(\mathbb{C})$ is an n_i ampliation of $M_{n_i}(\mathbb{C})$, and therefore has a separating vector.

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Each $M_{n_i}^{(n_i)}(\mathbb{C})$ has property a separating vector, so B has a separating vector.

Semi-Simple Algebras are Maximal P_1 Algebras

Theorem

Let $B \subset M_k(\mathbb{C})$ be a semi-simple algebra. If B has property P_1, then $dimB \leq k$. Furthermore, if $dimB = k$, then B is a maximal P_1 algebra.

This result supports the idea that if $B \subset M_n(\mathbb{C})$ is an algebra with property P_1, then $\dim B \leq k$.

Dimension

Conjecture

Let $A \subset M_n(\mathbb{C})$ be an algebra with property P_1. Then $\dim A \leq n$.

Conjecture

Let $A \subset M_n(\mathbb{C})$ be an algebra with property P_1. Then A has a separating vector.

So far, no counterexamples have been noticed.

Classifications of $M_n(\mathbb{C})$

In $M_2(\mathbb{C})$, all P_1 spaces have been classified. This work has not been carried on past $M_2(\mathbb{C})$.
We have started on $M_3(\mathbb{C})$ and $M_4(\mathbb{C})$ P_1 algebras.

Bases and Frames

Let $\{x_i\}_{i=1}^{n}$ be a basis for R^n. Let $y_i = x_i \otimes x_i$. Does the space $S = [y_i]_{i=1}^{n}$ have property P_1? In 2-dimensions, this is possible. Let $\{x_i\}_{i=1}^{k}$ be a frame for R^n, $n \leq k$. let $y_i = x_i \otimes x_i$ Let $S = [y_i]$ When does S have property P_1?

End

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